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Newton's Law of Cooling states that the rate of change of the surface temperature of an object is proportional to the difference between the temperature of the object and the temperature of its surrounding at that time.

The room temperature in a café is kept constant at 20°C by means of its central cooling system. It was found that the temperature of a cup of coffee left on a table in the café will drop from 60°C to 45°C in 10 minutes. Express this information as a differential equation connecting θ in degree Celsius and t in minutes, where θ is the temperature of the coffee at time t.

One day, Daniel bought a cup of "extra hot" coffee and noticed that it took 20 minutes for the temperature to cool to 45°C. What is the temperature of his coffee when he first bought it? **[6]**

By Newton's Law of Cooling,

$$\frac{d\theta}{dt} = k(\theta - 20)$$

$$\int \frac{d\theta}{\theta - 20} = \int k\, dt$$

$$\therefore \ \ln|\theta - 20| = kt + C \ \text{—} \ ①$$

Method 1

Given: when $t = t_1, \theta = 60$ — **A**

When $t = t_1 + 10, \theta = 45$ — **B**

From ①: $\ln 40 = kt_1 + C$, $\ln 25 = kt_1 + 10k + C$

$\Rightarrow \quad \ln\frac{25}{40} = 10k$

$\Rightarrow \quad k = \frac{1}{10}\ln\frac{5}{8}$ — ②

Also given: $t = t_2, \theta = ?$

$t = t_2 + 20, \theta = 45$ — **C**

From ① & ②: $\ln|\theta - 20| = \left(\frac{1}{10}\ln\frac{5}{8}\right)t_2 + C_1$,

$$\ln 25 = \left(\frac{1}{10}\ln\frac{5}{8}\right)t_2 + 20\left(\frac{1}{10}\ln\frac{5}{8}\right) + C_1$$

$\Rightarrow \quad \ln\frac{25}{\theta - 20} = 2\ln\frac{5}{8}$ [Modulus can be removed as initial temperature is higher than 45°C, hence $\theta - 20 > 0$.]

$\Rightarrow \quad \theta = \frac{25}{\left(\frac{5}{8}\right)^2} + 20 = 84\ °\text{C}$

Method 2

Given: $t = 0, \theta = 60$ — **A**

$t = 10, \theta = 45$ — **B**

From ①: $\ln 40 = C$, $\ln 25 = 10k + C$

$\Rightarrow \quad \ln\frac{25}{40} = 10k$

$\Rightarrow \quad k = \frac{1}{10}\ln\frac{5}{8}$ — ②

Also given: $t=0,\ \theta=?$

$t=20,\ \theta=45$ — **C**

From ① & ②: $\ln|\theta-20|=\left(\frac{1}{10}\ln\frac{5}{8}\right)t_2+C_1$,

$$\ln 25=20\left(\frac{1}{10}\ln\frac{5}{8}\right)+C_1$$

$\Rightarrow$ $\ln\frac{25}{\theta-20}=2\ln\frac{5}{8}$ [Modulus can be removed as initial temperature is higher than 45°C, hence $\theta-20>0$.]

$$\Rightarrow \quad \frac{25}{\theta-20}=\left(\frac{5}{8}\right)^2$$

$$\Rightarrow \quad \theta=\frac{25}{\left(5/8\right)^2}+20=84\ ^\circ\text{C}$$