**TJC/Promo 2006/6**

**6** A geometric progression has first term $a$ (where $a > 0$), and common ratio $r$. The sum of the first $n$ terms is $S_n$ and the sum to infinity is $S$. Given that $S_2$ is twice the value of the fifth term, find the value of $r$. Hence find the least value of $n$ such that $S_n$ is within 5% of $S$. [7]

**Solution**

Given that $2T_5 = S_2$, we have $2ar^4 = a + ar$,

i.e., $2r^4 - r - 1 = 0$.

From G.C., $r = -0.648$ or $r = 1$. (rejected as $|r| < 1$ for $S$ to exist).

For $S_n$ is within 5% of $S$, i.e., $|S_n - S| < 0.05S$,

we have $\left|\frac{a(1-r^n)}{1-r} - \frac{a}{1-r}\right| < \frac{0.05a}{1-r}$ --------------(1)

Since $\frac{a}{1-r} > 0$ (as $a > 0$ and $r = -0.648$), (1) simplifies to

$$\left|(1-(-0.648)^n)-1\right| < 0.05$$

$$(0.648)^n < 0.05$$

$$n\lg(0.648) < \lg(0.05)$$

$$n > \frac{\lg(0.05)}{\lg(0.648)} = 6.90 \text{ (correct to 3 significant figures)}$$

Hence, minimum value of $n$ is 7.



**N08/I/10**

**10** **(i)** A student saves \$10 on 1 January 2009. On the first day of each subsequent month she saves \$3 more than in the previous month, so that she saves \$13 on 1 February 2009, \$16 on 1 March 2009, and so on. On what date will she first have saved over \$2000 in total? [5]

**(ii)** A second student puts \$10 on 1 January 2009 into a bank account which pays compound interest at a rate of 2% per month on the last day of each month. She puts a further \$10 into the account on the first day of each subsequent month.

**(a)** How much compound interest has her original \$10 earned at the end of 2 years? [2]

**(b)** How much in total is in the account at the end of 2 years? [3]

**(c)** After how many complete months will the total in the account first exceed \$2000? [4]

**Solution**

(i) Observe the sequence of amounts (in $) saved:
10, 13, 16, …
This follows an arithmetic progression with first term 10 and common difference 3.

Now, $S_n > 2000$ where $n$ represents the number of terms.
[ Use the data from the question: "saved over $2000 in total". ]

$$\Rightarrow \frac{n}{2}\left[2(10)+(n-1)(3)\right] > 2000$$

$$\Rightarrow 10n + \frac{3}{2}n^2 - \frac{3}{2}n > 2000$$

$$\Rightarrow \frac{3}{2}n^2 + \frac{17}{2}n - 2000 > 0$$

For $\frac{3}{2}n^2 + \frac{17}{2}n - 2000 = 0$, $n = \dfrac{-\frac{17}{2} \pm \sqrt{\left(\frac{17}{2}\right)^2 - 4\left(\frac{3}{2}\right)(-2000)}}{2\left(\frac{3}{2}\right)}$

$= -39.5$ or $33.8$

[ Recall that $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ are the roots of $ax^2 + bx + c = 0$. ]

For our inequality, $n < -39.5$ or $n > 33.8$
$\Rightarrow n > 33.8$, since $n > 0$
Least $n = 34$
Date is 2011, 1 October. ❖
[ 34 months is equivalent to 2 years and 10 months. ]

(ii)(a) Interest earned $= 10(1.02)^{24} - 10$
$= \$6.08$ (to 3 s. f.) ❖

[ Principal with interest compounded over time is given by $p\left(1+\dfrac{r}{100}\right)^t$ where $p$ is the principal amount, $r$ is the interest rate per month and $t$ is the time in months.

We need to exclude the principal amount to give us the interest earned. ]

(b) In month 1, start: 10, **end**: $1.02(10)$
[ Geometric series of 1 term. ]

In month 2, start: $1.02(10)+10$, **end**: $1.02^2(10)+1.02(10)$
[Geometric series of 2 terms. ]

In month 3, start: $1.02^2(10)+1.02(10)+10$, **end**: $1.02^3(10)+1.02^2(10)+1.02(10)$
[ Geometric series of 3 terms. ]

In month 24, we will **end** with a total of
$1.02^{24}(10)+1.02^{23}(10)+...+1.02^2(10)+1.02(10)$
[ Geometric series of 24 terms, following the pattern we saw above. ]

$$=10\frac{1.02\left(1.02^{24}-1\right)}{1.02-1}$$
$= \$310$ (to 3 s. f.) ❖

(c) Let $n$ be the number of complete months.

$10\frac{1.02\left(1.02^{n}-1\right)}{1.02-1}>2000$ [ Use the data from the question: "total…first exceed \$2000". ]

$\Rightarrow 1.02^n > \frac{2000}{510}+1$

$\Rightarrow n\ln 1.02 > \ln\left(\frac{200}{51}+1\right)$ [ Take natural logarithm of both sides. ]

$\Rightarrow n > 80.5$

$\therefore$ total exceeds \$2000 after 81 complete months. ❖