Specimen Paper/2007/1/8

The positive numbers x_n satisfy the relation

$$x_{n+1} = \sqrt{x_n + 5}, \text{ for } n = 1, 2, 3, \ldots .$$

As $n \to \infty$, $x_n \to l$.

(i) Find (in either order) the value of l to 3 decimal places and the exact value of l.

(ii) Prove that $(x_{n+1})^2 - l^2 = x_n - l$.

(iii) Hence, show that if $x_n > l$, $x_n > x_{n+1} > l$.

Solution

(i) Since when $n \to \propto$, $x_n \to l$ and also $x_{n+1} \to l$

Hence, the recurrence relation $x_{n+1} = (x_n + 5)^{1/2}$ tends to $l = (l+5)^{1/2}$

$$\Rightarrow \quad l^2 - (l+5) = 0$$

$$\Rightarrow \quad l^2 - l - 5 = 0$$

$$\Rightarrow \quad l = \frac{1 \pm \sqrt{1 - 4(-5)}}{2}$$

$$\Rightarrow \quad l = \frac{1 \pm \sqrt{21}}{2} \text{ or } 2.791 \text{ (3 d.p.)}$$

(ii) $(x_{n+1})^2 - l^2 = (x_n + 5) - (l + 5) .= x_n - l$ (proved)

(iii) $x_n > l \quad \Rightarrow \quad x_n - l > 0.$

$$\Rightarrow \quad (x_{n+1})^2 - l^2 > 0$$

$$\Rightarrow \quad (x_{n+1} + l)(x_{n+1} - l) > 0$$

$$\Rightarrow \quad (x_{n+1} - l) > 0 \quad \because (x_{n+1} - l) > 0$$

$$\Rightarrow \quad x_{n+1} > l$$

$$(x_{n+1})^2 - l^2 = x_n - l$$

$$\Rightarrow (x_{n+1} + l)(x_{n+1} - l) = x_n - l$$

$$\Rightarrow (x_{n+1} - l) = \frac{x_n - l}{x_{n+1} + l} < x_n - l \qquad (\because (x_{n+1} + l) > 1)$$

$$\Rightarrow x_{n+1} < x_n$$

$$\therefore x_n > x_{n+1} > l$$

MJC/Prelim2010/1/3

A sequence of positive real numbers $x_1, x_2, x_3, \ldots$ satisfies the recurrence relation

$$x_{n+1} = \frac{3x_n}{\sqrt{2+x_n}}$$

for $n \geq 1$.

(i) Given that as $n \to \infty$, $x_n \to \alpha$, find the exact value of α. [2]

(ii) Show that this sequence is

(a) strictly increasing for $0 < x_n < \alpha$, [2]

(b) strictly decreasing for $x_n > \alpha$. [2]

Solution

(i)

As $n \to \infty, x_n \to \alpha$ & $x_{n+1} \to \alpha$

$\Rightarrow \alpha = \dfrac{3\alpha}{\sqrt{2+\alpha}} \Rightarrow \alpha^2 = \dfrac{9\alpha^2}{2+\alpha}$

$\Rightarrow \alpha^2(2+\alpha) - 9\alpha^2 = 0$

$\Rightarrow \alpha^2(\alpha - 7) = 0 \Rightarrow \alpha = 0$ or $\alpha = 7$.

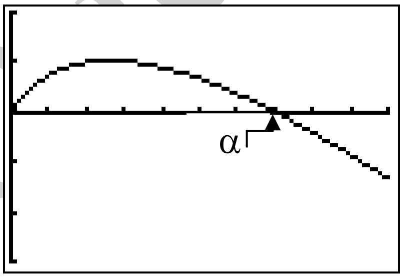
Thus $\alpha = 7$ since $\alpha > 0$.

(ii)

Method 1 [Graphical]

$x_{n+1} - x_n = \dfrac{3x_n}{\sqrt{2+x_n}} - x_n$

Sketch $y = x_{n+1} - x_n = \dfrac{3x_n}{\sqrt{2+x_n}} - x_n$



From the graph:

For $0 < x_n < \alpha$: $x_{n+1} - x_n > 0 \Rightarrow x_{n+1} > x_n$

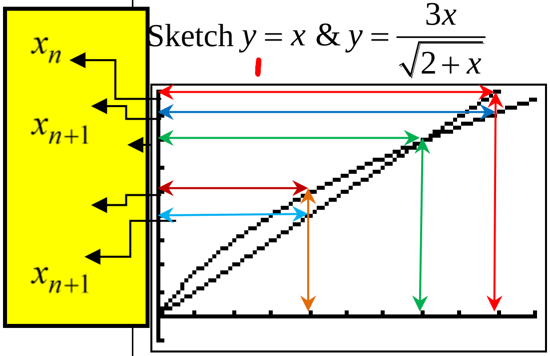
Therefore the sequence is strictly increasing.

For $x_n > \alpha$: $x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n$

Therefore the sequence is strictly decreasing. [Shown]

Method 2 [Graphical]

Sketch $y = x$ & $y = \dfrac{3x}{\sqrt{2+x}}$



From the graph:

For $0 < x_n < \alpha$: $x_{n+1} > x_n$

Therefore the sequence is strictly increasing.

For $x_n > \alpha : x_{n+1} < x_n$

Therefore the sequence is strictly decreasing. [Shown]

Method 3 [Algebraic]

$$x_{n+1} - x_n = \frac{3x_n}{\sqrt{2+x_n}} - x_n = \frac{3x_n - x_n\sqrt{2+x_n}}{\sqrt{2+x_n}}$$

$$= \frac{x_n\left(3-\sqrt{2+x_n}\right)}{\sqrt{2+x_n}}$$

For $0 < x_n < \alpha$:

$\sqrt{2+x_n} > 0, x_n > 0$ & $3-\sqrt{2+x_n} > 0$ since $\sqrt{2+x_n} < \sqrt{9} = 3$

Thus: $x_{n+1} - x_n > 0 \Rightarrow x_{n+1} > x_n$

Therefore the sequence is strictly increasing.

For $x_n > \alpha$

$\sqrt{2+x_n} > 0, x_n > 0$ & $3-\sqrt{2+x_n} < 0$ since $\sqrt{2+x_n} > \sqrt{9} = 3$

Thus: $x_{n+1} - x_n < 0 \Rightarrow x_{n+1} < x_n$

Therefore the sequence is strictly decreasing. [Shown]