AJC/Mid-yr CT 2010/Q14

With reference to a fixed origin O, the points P and Q have position vectors $3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $5\underset{\sim}{i} + 3\underset{\sim}{j} + \alpha\underset{\sim}{k}$, where $\alpha \in \mathbb{Z}$. The line ℓ has equation $\underset{\sim}{r} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} \beta \\ -1 \\ 1 \end{pmatrix}$, $\lambda \in \Re$ and the plane Π_1 has Cartesian equation $3z - 4y = 10$.

(i) Given that the shortest distance from Q to the plane Π_1 is 1 unit. Find the value of α and determine, with reason, whether Q and the origin O are on the same side or on opposite sides of Π_1. [3]

(ii) If the maximum angle between line ℓ and plane Π_1 is $\frac{\pi}{6}$. Find the range of values of β. [3]

(iii) Find the Cartesian equation of the plane Π_2 that passes through point P and contains the line ℓ when $\beta = 1$. Hence describe, with reasons, the geometrical relationship of the three planes Π_1, Π_2 and Π_3, where plane Π_3 has Cartesian equation $11x - 3y - 28z = 1$. [4]

Π_1: $\quad \underset{\sim}{r} \bullet \begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = 10 \quad \Rightarrow \quad \underset{\sim}{r} \bullet \frac{1}{5}\begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} = \frac{10}{5} = 2$

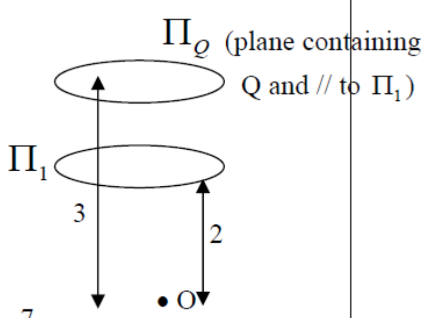
(i) Distance from Q to Π_1 $= \left| \begin{pmatrix} 5 \\ 3 \\ \alpha \end{pmatrix} \bullet \frac{1}{5}\begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} - 2 \right| = \left| \frac{3\alpha - 12}{5} - 2 \right| = \left| \frac{3\alpha - 22}{5} \right|$

$\Rightarrow \left| \frac{3\alpha - 22}{5} \right| = 1 \Rightarrow \quad 3\alpha - 22 = \pm 5 \quad \Rightarrow \quad \alpha = 9$ or $17/3$ (rejected as α is an integer)

Q and O are on opposite sides of the plane

[Reason: distance from Q to O = $\begin{pmatrix} 5 \\ 3 \\ \alpha \end{pmatrix} \bullet \frac{1}{5}\begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} \frac{3\alpha - 12}{5} = 3$

and distance from Π_1 to O is 2]



(ii) Angle between Π_1 and ℓ is $= \sin^{-1} \left| \frac{\begin{pmatrix} 0 \\ -4 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} \beta \\ -1 \\ 1 \end{pmatrix}}{5 \cdot \sqrt{2 + \beta^2}} \right| = \sin^{-1} \frac{7}{5\sqrt{2 + \beta^2}}$

As $\sin\frac{\pi}{6} = \frac{1}{2}$ and sine function is increasing in 1[st] quadrant,

$\frac{7}{5\sqrt{2+\beta^2}} \le \frac{1}{2} \quad \Rightarrow \beta^2 \ge \frac{146}{25} \quad \Rightarrow \beta \ge \frac{\sqrt{146}}{5} (= 2.416609..)$ or $\beta \le -\frac{\sqrt{146}}{5}$

(iii)Vector // to $\Pi_2 = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

Normal vector of Π_2 is $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$

Equation of Π_2 is $\underset{\sim}{r} \bullet \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = -6$

Hence Cartesian equation of the plane Π_2 is $-x + y + 2z = -6$.

Π_1 : $0 - 4y + 3z = 10$

Π_2 : $-x + y + 2z = -6$

Π_3 : $11x - 3y - 28z = 1$

Augmented matrix $A = \left(\begin{array}{ccc|c} 0 & -4 & 3 & 10 \\ -1 & 1 & 2 & -6 \\ 11 & -3 & -28 & 1 \end{array}\right)$ rref(A) = $\left(\begin{array}{ccc|c} 1 & 0 & -11/4 & 0 \\ 0 & 1 & -3/4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$

Since the last equation is inconsistent, therefore the system of equations has no solutions.
Geometrical interpretation: the 3 planes have no common point. And since their normals are not parallel, the 3 planes form a triangular prism.