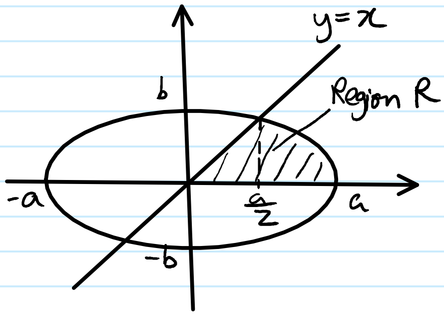
Application of Integration

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = \frac{b^2}{a^2}\left(a^2 - x^2\right)$



$$\text{Area } A = 4\int_0^a y\,dx$$
$$= 4\int_a^b \frac{b}{a}\sqrt{a^2-x^2}\,dx$$
$$= \frac{4b}{a}\int_0^a \sqrt{a^2-x^2}\,dx \text{ \# (shown)}$$

Let $x = a\sin\theta \Rightarrow \frac{dx}{d\theta} = a\cos\theta$

When $x = a$, $\theta = \frac{\pi}{2}$
" $x = 0$, $\theta = 0$

$$\text{Area } A = \frac{4b}{a}\int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta\,d\theta)$$
$$= 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 4ab\int_0^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta$$
$$= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}} = 2ab\left[\frac{\pi}{2} - 0\right]$$
$$= \pi ab \text{ unit}^2$$

When $a^2 = 3b^2$, $y^2 = \frac{b^2}{a^2}(a^2 - x^2) = \frac{1}{3}(a^2 - x^2)$

When $y = x$, $x^2 = \frac{1}{3}(a^2 - x^2) \Rightarrow 4x^2 = a^2 \Rightarrow x = \pm\frac{a}{2}$

$$\text{Area of } R = \text{Area of } \Delta + \int_{\frac{a}{2}}^{a} \sqrt{\tfrac{1}{3}(a^2-x^2)}\,dx$$
$$= \frac{1}{2}\left(\frac{a}{2}\right)\left(\frac{a}{2}\right) + \frac{1}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta\,d\theta)$$
$$= \frac{a^2}{8} + \frac{a^2}{\sqrt{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,d\theta$$
$$= \frac{a^2}{8} + \frac{a^2}{2\sqrt{3}}\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$
$$= \frac{a^2}{8} + \frac{\sqrt{3}a^2}{6}\left[\frac{\pi}{2} - \left(\frac{\pi}{6} + \frac{\sqrt{3}}{4}\right)\right]$$
$$= \frac{a^2}{8} + \frac{\sqrt{3}a^2}{6}\left[\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right]$$
$$= \frac{a^2}{8} + \frac{\sqrt{3}a^2\pi}{18} - \frac{a^2}{8} = \frac{\sqrt{3}a^2\pi}{18} \text{ \#}$$

2(i) $x = t^2$, $y = t^3 \Rightarrow \frac{dx}{dt} = 2t$, $\frac{dy}{dt} = 3t^2$

$\frac{dy}{dx} = \frac{3t^2}{2t} = \frac{3}{2}t$

When $t = 1$, $x = 1$, $y = 1$, $\frac{dy}{dx} = \frac{3}{2}$

Equation of normal is $y - 1 = -\frac{2}{3}(x - 1)$

$\therefore y = -\frac{2}{3}x + \frac{5}{3}$

When $y=0$, $\frac{2}{3}x = \frac{5}{3} \Rightarrow x = \frac{5}{2}$

(ii) Required Area $= \int_0^1 y\,dx + \text{Area of } \triangle$

$= \int_0^1 t^3 (2t\,dt) + \frac{1}{2}\left(\frac{5}{2}-1\right)(1)$

$= 2\left[\frac{t^5}{5}\right]_0^1 + \frac{3}{4} = 2\left(\frac{1}{5}\right) + \frac{3}{4} = \frac{23}{20}$ unit^2 #

3. $\int (\ln x)^2\,dx$, Sub $\ln x = y \Rightarrow x = e^y$

$\frac{dx}{dy} = e^y$

$= \int y^2 (e^y\,dy) = \int y^2 e^y\,dy$ # (shown)

$= y^2(e^y) - \int e^y (2y)\,dy$

$= e^y y^2 - 2\left[ye^y - \int e^y\,dy\right] = e^y y^2 - 2ye^y + 2e^y + c$

$= x(\ln x)^2 - 2(\ln x)x + 2x + c = x\left[(\ln x)^2 - 2\ln x + 2\right] + c$ # (shown)

(a) Required Volume $= \pi \int_1^e y^2\,dx = \pi \int_1^e (\ln x)^2\,dx$

$= \pi\left[x\left[(\ln x)^2 - 2\ln x + 2\right]\right]_1^e = \pi\left[(e-2+2)-2\right] = \pi(e-2)$ unit^3 #

(b) Required volume $= \pi\int_0^1 x^2\,dy = \pi\int_0^1 (e^y)^2\,dy$

$= \pi\left[\frac{e^{2y}}{2}\right]_0^1 = \pi\left[\frac{e^2}{2} - \frac{1}{2}\right] = \frac{\pi}{2}(e^2-1)$ unit^3 #