Complex Numbers

1.

$$w + z = 6 + 2i \quad -①$$

$$i(w - 3z) = \frac{20}{2-i} \times \frac{2+i}{2+i} = \frac{{}^{4}\cancel{20}(2+i)}{\cancel{5}}$$

$① - ②$:

$$4z = 6 + 2i - \frac{20}{2-i} \times \frac{2+i}{2+i}$$

$$= 6 + 2i - \frac{20(2+i)}{4+1}$$

$$= 6 + 2i - 4(2+i)$$

$$z = \frac{-2 - 2i}{4}$$

$$= -\frac{1}{2}(1+i) = -\frac{1}{2} - \frac{1}{2}i \ \#$$

$$w = 6 + 2i + \frac{1}{2}(1+i) = \frac{13}{2} + \frac{5}{2}i \ \#$$

3.

$$(3-i)^2 = 9 - 6i - 1$$

$$= 8 - 6i$$

$$(z+i)^2 = -(8 + 6i)$$

$$= -(3-i)^2$$

$$(z+i)^2 = (i)^2(3-i)^2$$

$$= (+1 + 3i)^2$$

$$z + i = \pm(+1 + 3i)$$

$$z = +1 + 3i - i \text{ or } -1 - 3i - i$$

$$= +1 + 2i \ \# \text{ or } -1 - 4i \ \#$$

2.

$$\frac{z}{z^*} = \frac{3}{5} + \frac{4}{5}i \quad -①$$

Sub $z^* = \frac{5}{z}$ into ①:

$$\frac{z^2}{5} = \frac{3+4i}{5}$$

$$(x+yi)^2 = 3 + 4i$$

$$x^2 - y^2 = 3 \ \& \ 2xy = 4$$

$$x^2 - \frac{4}{x^2} = 3 \qquad y = \frac{2}{x}$$

$$x^4 - 3x^2 - 4 = 0$$

$$(x^2 - 4)(x^2 + 1) = 0$$

$$x^2 = 4 \text{ or } x^2 = -1$$

$$x = \pm 2 \Rightarrow y = \pm 1$$

$$z = \pm(2+i) \ \#$$

4(a)

$$1 + e^{\frac{\pi}{3}i} = e^{i(\frac{\pi}{6})}\left(e^{-i(\frac{\pi}{6})} + e^{i(\frac{\pi}{6})}\right)$$

$$= e^{i(\frac{\pi}{6})}\left(\cos\frac{\pi}{6} + \cancel{i\sin\frac{\pi}{6}} + \cos\frac{\pi}{6} - \cancel{i\sin\frac{\pi}{6}}\right)$$

$$= 2\cos\frac{\pi}{6}\, e^{i(\frac{\pi}{6})}$$

$$\Rightarrow r = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3} \ \# \text{ (shown)}$$

$$\arg\left(1 + e^{\frac{\pi}{3}i}\right) = \frac{\pi}{6} \ \#$$

OR

$$1 + \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = 1 + \frac{1}{2} + i\left(\frac{\sqrt{3}}{2}\right)$$

$$r = \sqrt{\left(\frac{3}{2}\right)^2 + \frac{3}{4}} = \sqrt{3}$$

$$\cancel{1} + \left(2\cos^2\frac{\pi}{6} - \cancel{1}\right) + 2i\sin\frac{\pi}{6}\cos\frac{\pi}{6}$$

$$= 2\cos\frac{\pi}{6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

4(b) $e^{+\left(\frac{5}{4}\pi - \frac{\pi}{3}\right)i} = e^{\frac{5\pi}{12}i} = \cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}$

$= 0.259 + i(0.966)$ # (3 s.f.)

(c) $e^{1+20i} = 1.11 + 2.48i$

$e^1 \cdot e^{20i} = e(\cos 20 + i\sin 20)$

$\cos(1.150444078)$

5(a) $|-7+24i| = \sqrt{7^2+24^2} = \sqrt{625} = 25$ #

$\arg(-7+24i) = \tan^{-1}\left(\frac{24}{-7}\right) = 1.85$ #

$\operatorname{Im}(-7+24i)^8 = \operatorname{Im}\left[25(\cos 1.85459 + i\sin(1.85459)\right]^8$

$= \operatorname{Im} 25^8\left[\cos 8(1.85459) + i\sin 8(1.85459)\right]$

$= 1.168492359 \times 10^{11} \approx 1.17 \times 10^{11}$ #

(b) $\left|\frac{1+i}{\sqrt{2}}\right| = \frac{\sqrt{2}}{\sqrt{2}} = 1$

$\arg(1+i) - \arg\sqrt{2} = \frac{\pi}{4}$

$\frac{1+i}{\sqrt{2}} = e^{i\left(\frac{\pi}{4}\right)}$

$\left(e^{i\frac{\pi}{4}}\right)^7 + \left(e^{i\left(-\frac{\pi}{4}\right)}\right)^7 = 2\cos 7\left(\frac{\pi}{4}\right) = 1.414213562$

$= 2\cos\frac{7\pi}{4} = 2\cos\left(\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}$ #