Solution for DE Extra Question

The variables *x* and *y* are related by the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x}-y+\frac{2}{x}=0$.

Show that by means of the substitution $y=ux$, the differential equation may be reduced to

$\frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{2}{x^3}$.

Hence, or otherwise, find the general solution for *y* in terms of *x*.

Sketch, for $x>0$, two members of the family of solution curves corresponding to a positive value and a negative value of the arbitrary constant.

Solution

Given $x\frac{\mathrm{d}y}{\mathrm{d}x}-y+\frac{2}{x}=0\ldots(1)$

Let $y=ux$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$$

Sub into eq (1)

$$x\left(u+x\frac{\mathrm{d}u}{\mathrm{d}x}\right)-ux+\frac{2}{x}=0$$

$$xu+x^2\frac{\mathrm{d}u}{\mathrm{d}x}-ux+\frac{2}{x}=0$$

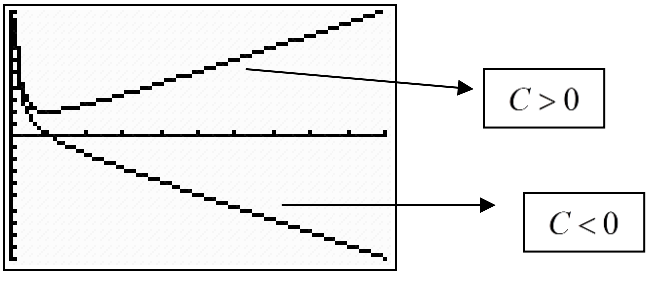
$$\frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{2}{x^3}\text{ (Shown)}$$

$$u=\int-\frac{2}{x^3}\mathrm{d}x$$

$$u=-2\left[\frac{x^{-3+1}}{-2}\right]+C$$

$$\frac{y}{x}=\frac{1}{x^2}+C$$

$$y=\frac{1}{x}+Cx$$



The rate at which a chemical evaporates at room temperature is proportional to the amount of substance which has not yet evaporated. If the initial amount of chemical was 500g and the amount which has evaporated at time t is x. Write down a differential equation involving x and t. Show that $x = 500 - Ae^{-kt}$, where A and k are constants.

Show that the time taken for half the chemical to evaporate is $\frac{1}{k}\ln 2$.

Solution

$$\frac{dx}{dt} = a(500 - x)$$

$$\int \frac{1}{500 - x} dx = \int a \, dt$$

$$-\ln|500 - x| = at + C$$

$$\ln|500 - x| = -at - C$$

$$500 - x = e^{-at-C}$$

$$x = 500 - Ae^{-at}, A = e^{-C}$$

$x = 500 - Ae^{-kt}, k = a$ (Shown)

When $t = 0, x = 0$

$$0 = 500 - A$$

A=500

$$\therefore x = 500 - 500e^{-kt}$$

When x =250 (Half the chemical)

$$250 = 500 - 500e^{-kt}$$

$$500e^{-kt} = 250$$

$$e^{-kt} = \frac{250}{500} = \frac{1}{2}$$

$$-kt = \ln\frac{1}{2}$$

$$t = -\frac{1}{k}\ln\frac{1}{2}$$

$t = \frac{1}{k}\ln 2$ (shown)