Assignment: Differential Equations

1 (i)

$$\frac{dy}{dx} = \sqrt{\left(\frac{y}{x}\right)}$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

$$\int y^{-\frac{1}{2}}\,dy = \int x^{-\frac{1}{2}}\,dx \qquad [1]$$

$2\sqrt{y} = 2\sqrt{x} + c$, where c is an arbitrary constant [1]

$$\sqrt{y} = \sqrt{x} + \frac{c}{2}$$

General Solution: $y = \left(\sqrt{x} + \frac{c}{2}\right)^2$ [1]

(ii)

When $x = 1$, $y = 4$,

$$2\sqrt{4} = 2\sqrt{1} + c$$

$c = 2$ [1]

Particular Solution: $y = \left(\sqrt{x} + 1\right)^2$

$$y = x + 2\sqrt{x} + 1$$

$$\frac{dy}{dx} = 1 + x^{-\frac{1}{2}} \qquad [1]$$

$$\frac{dy}{dx} = 1 + \frac{1}{\sqrt{x}} > 1 \text{ for all } x > 0 \qquad [1]$$

2.

$$u = xy$$

$$\frac{du}{dx} = x\frac{dy}{dx} + y \qquad [1]$$

$$x^2\frac{dy}{dx} + xy - y^2 = 0$$

$$x\left(x\frac{dy}{dx}\right) + xy - \left(\frac{u}{x}\right)^2 = 0 \qquad [1]$$

$$x\left(\frac{du}{dx} - y\right) + xy = \frac{u^2}{x^2}$$

$$x\frac{du}{dx} = \frac{u^2}{x^2}$$

$$x^3\frac{du}{dx} = u^2 \qquad [1]$$

	$\frac{1}{u^2}\frac{du}{dx} = \frac{1}{x^3}$ $\int u^{-2}\,du = \int x^{-3}\,dx$ [1] **General Solution:** $-\frac{1}{u} = -\frac{1}{2x^2} + c$, where c is an arbitrary constant [1] When $x = 1, y = 1$ $-1 = -\frac{1}{2} + c$ $c = -\frac{1}{2}$ [1] $-\frac{1}{u} = -\frac{1}{2x^2} - \frac{1}{2}$ $\frac{1}{u} = \frac{1}{2x^2} + \frac{1}{2}$ [1] $\frac{1}{u} = \frac{1+x^2}{2x^2}$ $u = \frac{2x^2}{1+x^2}$ $xy = \frac{2x^2}{1+x^2}$ **Particular Solution:** $y = \frac{2x}{1+x^2}$ [1]

3 (i)	$\frac{dx}{dt} = \text{rate of birth} - \text{rate of death} = 2x - ax^2$ At $x = 10$, $0 = 2(10) - a(10)^2 \Rightarrow a = 0.2$ $\therefore \frac{dx}{dt} = 2x - \frac{x^2}{5}$
(ii)	Selling away 1800 prawns daily, D.E. becomes $\frac{dx}{dt} = 2x - \frac{x^2}{5} - 1.8$ $\frac{dx}{dt} = 2x - \frac{x^2}{5} - \frac{9}{5}$ $\frac{dx}{dt} = -\frac{1}{5}\left(x^2 - 10x + 9\right) = -\frac{1}{5}(x-9)(x-1)$
	Separating the variables, $\frac{1}{(x-9)(x-1)}\frac{dx}{dt} = -\frac{1}{5}$

	Integrating both sides w.r.t. t: $\int \frac{1}{(x-9)(x-1)}\,\mathrm{d}x = \int -\frac{1}{5}\,\mathrm{d}t$ $\int \frac{1}{8(x\text{-}9)} - \frac{1}{8(x-1)}\mathrm{d}x = \int -\frac{1}{5}\,\mathrm{d}t$ $ln(x-9) - ln(x-1) = -\frac{8t}{5} + c,$ where c is an arbitrary constant $\frac{x-9}{x-1} = Ae^{-\frac{8t}{5}},\ A = e^{c}$ At $t = 0$, $x = 13$: $A = \frac{13-9}{13-1} = \frac{1}{3}$ $\therefore \frac{x-9}{x-1} = \frac{e^{-\frac{8t}{5}}}{3} \Rightarrow x = \frac{24}{3 - e^{-\frac{8t}{5}}} + 1$ In the long term: as $t \to \infty$, $e^{-\frac{8t}{5}} \to 0$. $\therefore x = \frac{24}{3 - e^{-\frac{8t}{5}}} + 1 \to 9$ Hence, the population of prawns will stabilise at 9000 in the long term.