Integration and Applications - Solutions

1	(i) $\int x^2 e^{2x^3+5}dx = \frac{1}{6}e^{2x^3+5} + c$ $\int x^5 e^{2x^3+5}dx$ (ii) $= \int x^3\left(x^2 e^{2x^3+5}\right)dx$ $= \left[\frac{1}{6}e^{2x^3+5}\right] - \int 3x^2\left[\frac{1}{6}e^{2x^3+5}\right]dx$ $= \frac{1}{6}x^3 e^{2x^3+5} - \frac{1}{12}e^{2x^3+5} + c$ or $\frac{1}{12}e^{2x^3+5}\left(2x^3 - 1\right) + c$

2(a) Let $\sqrt{x} = \cos\theta$

Differentiate with respect to θ:

$$\frac{1}{2\sqrt{x}}\frac{dx}{d\theta} = -\sin\theta \Rightarrow dx = -2\sqrt{x}\ \sin\theta\, d\theta = -2\cos\theta\sin\theta\, d\theta$$

$$\int\sqrt{\frac{x}{1-x}}\,dx = \int\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}(-2\cos\theta\sin\theta)\,d\theta$$

$$= -\int\frac{\cos\theta}{\sin\theta}[2\cos\theta\sin\theta]\,d\theta$$

$$= -\int 2\cos^2\theta\, d\theta$$

$$= -\int \cos 2\theta + 1 \quad d\theta$$

$$= -\frac{1}{2}\sin 2\theta - \theta + C$$

$$= -\sin\theta\cos\theta - \theta + C$$

$$= -\sqrt{1-\cos^2\theta}\ \cos\theta - \theta + C$$

$$= -\sqrt{1-x}\sqrt{x} - \cos^{-1}\sqrt{x} + C$$

$$= -\sqrt{x(1-x)} - \cos^{-1}\sqrt{x} + C$$

(b) Let $u = (\ln x)^2$ and $dv = x$

$du = 2(\ln x)(\frac{1}{x})$ and $v = \frac{1}{2}x^2$

$$\int_1^e x(\ln x)^2\,dx = [\tfrac{1}{2}x^2(\ln x)^2]_1^e - \int_1^e x(\ln x)\,dx$$

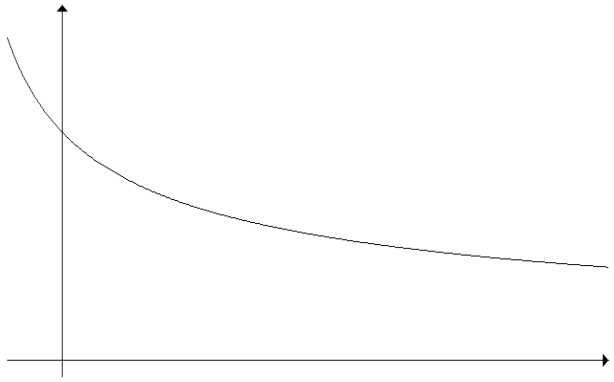
Let $u = \ln x$ and $dv = x$

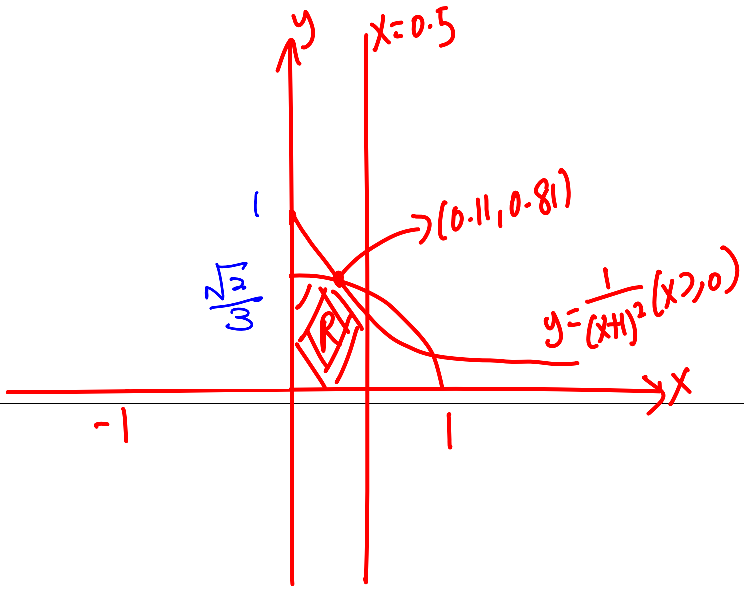
$du = \frac{1}{x}$ and $v = \frac{1}{2}x^2$

$$\int_1^e x(\ln x)^2\,dx = \frac{1}{2}e^2 - \{[\tfrac{1}{2}x^2\ln x]_1^e - \int_1^e \tfrac{1}{2}x\,dx\}$$

$$= \frac{1}{2}e^2 - \frac{1}{2}e^2 + [\tfrac{1}{4}x^2]_1^e$$

$$= \frac{1}{4}(e^2 - 1)$$

3(i)	
(ii)	Required area $= \int_{-1}^{0} y \, \mathrm{d}x = \int_{0}^{1} y \frac{\mathrm{d}x}{\mathrm{d}t} \, \mathrm{d}t$ $= \int_{0}^{1} \frac{1}{\sqrt{1+t^2}} (2\mathrm{t}) \, \mathrm{dt}$ $= 2\int_{0}^{1} \frac{t}{\sqrt{1+t^2}} \, \mathrm{dt}$ $= 2\int_{0}^{\frac{\pi}{4}} \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \sec^2\theta \, \mathrm{d}\theta$ $= 2\int_{0}^{\frac{\pi}{4}} \frac{\tan\theta}{\sqrt{\sec^2\theta}} \sec^2\theta \, \mathrm{d}\theta$ $= 2\int_{0}^{\frac{\pi}{4}} \sec\theta \tan\theta \, \mathrm{d}\theta$ $= 2[\sec\theta]_{0}^{\frac{\pi}{4}}$ $= 2\left(\sqrt{2}-1\right)$ units^2 $t = \tan\theta$ $\frac{\mathrm{d}t}{\mathrm{d}\theta} = \sec^2\theta$ When $t = 0,\ \theta = 0.$ When $t = 1,\ \theta = \frac{\pi}{4}$

4	

	The x-coordinates of the points of intersection are 0.110 and 0.946. $\text{Volume (R)} = \pi\int_0^{0.11}\left(\sqrt{\frac{2}{3}\left(1-x^2\right)}\right)^2 \text{dx} + \pi\int_{0.11}^{0.5}\frac{1}{(\text{x}+1)^4}\text{dx}$ $= 0.685 \text{ units}^3$ $\text{Volume (S)} = \pi\int_0^{\sqrt{\frac{2}{3}}} 1-\tfrac{3}{2}y^2\text{dy}$ $= \pi\left[y-\tfrac{1}{2}y^3\right]_0^{\sqrt{\frac{2}{3}}}$ $= \pi\left[\sqrt{\tfrac{2}{3}}-\tfrac{1}{3}\sqrt{\tfrac{2}{3}}\right]$ $= \pi\,\tfrac{2}{3}\sqrt{\tfrac{2}{3}} \text{ units}^3$