AP and GP solutions

1)

Let *d* be the common difference of the arithmetic progression.

u_1, u_4 and u_8 are in geometric progression, we have $\frac{u_4}{u_1} = \frac{u_8}{u_4}$

ie. $u_4^2 = u_1 u_8$

$$\left(u_1 + 3d\right)^2 = u_1\left(u_1 + 7d\right)$$

$$u_1^2 + 6u_1 d + 9d^2 = u_1^2 + 7u_1 d$$

$$u_1 d - 9d^2 = 0$$

$$d\left(u_1 - 9d\right) = 0$$

$d = 0$ (rejected, since given A.P. is increasing)

or $u_1 = 9d$ -------(1)

Also $u_{10} + u_{12} + u_{14} + ... + u_{38} + u_{40} = 1056$

$$\frac{16}{2}\left(u_{10} + u_{40}\right) = 1056$$

$$\left(u_1 + 9d\right) + \left(u_1 + 39d\right) = 132$$

$u_1 + 24d = 66$------(2)

Substitute (1) into (2):

$$33d = 66$$

$$d = 2$$

$$\Rightarrow u_1 = 18$$

$u_{108} = u_1 + 107d = 18 + 107(2) = 232$(shown)

2)

13a) $S_\infty - S_n \le 0.3$

$$\frac{3}{1-7/8} - \frac{3\left[1-\left(\frac{7}{8}\right)^n\right]}{1-7/8} \le 0.3$$

$n\ln\left(\frac{7}{8}\right) \le \ln 0.0125$

$n \ge \frac{\ln 0.0125}{\ln(7/8)}$ $\Rightarrow \left(\frac{7}{8}\right)^n \le 0.0125$

$n \ge 32.8$

$\Rightarrow$ least $n = 33$

(b) $S_n = S_{3n} - S_n$

$2S_n = S_{3n}$

$\therefore 2 \cdot \frac{n}{2} \cdot [2a + (n-1)(-2)] = \frac{3n}{2}[2a + (3n-1)(-2)]$

$a = 7n - 1$

3(a)	The sum of the first 3 terms is half the sum of the first 9 terms, hence $\frac{3(2a+2d)}{2} = \frac{1}{2}\frac{9(2a+8d)}{2}$ $\Rightarrow d = -\frac{a}{10}$
(b)(i)	$S_{n-1} = (a-2)^{-n+1} - 1$ Hence $u_n = S_n - S_{n-1} = (a-2)^{-n} - (a-2)^{-n+1} = (3-a)(a-2)^{-n}$. $\frac{u_n}{u_{n-1}} = \frac{(3-a)(a-2)^{-n}}{(3-a)(a-2)^{-n+1}} = \frac{1}{a-2}$, which is a constant. Thus, the sequence is a GP with common ratio $\frac{1}{a-2}$.
(b)(ii)	For the series to be convergent, $\lvert r \rvert < 1$, i.e. $\left\lvert \frac{1}{a-2} \right\rvert < 1$ $\lvert a-2 \rvert > 1$ $a-2 > 1$ or $a-2 < -1$ $\Rightarrow a > 3$ or $a < 1$ Hence the set of values of a is $\{a \in \mathbb{R} : a > 3 \text{ or } a < 1\}$

4)

Let *a* and *r* be the first term and common ratio of the arithmetic and geometric progression respectively.

$(a-2)+(4r)=-4 \Rightarrow a=-2-4r$ ------------- (1)

$(a+4)+(a-2+4r)+(a-4+4r^2)=-4$

$\Rightarrow (a+4)-4+(a-2(2)+4(r^2))=-4$

$\Rightarrow a+2r^2=0$ --------------- (2)

Sub (1) into (2),

$(-2-4r)+2r^2=0$

$2r^2-4r-2=0$

$r^2-2r-1=0$

$r=\frac{2\pm 2\sqrt{2}}{2}=1\pm\sqrt{2}$

$\therefore r=1-\sqrt{2}$ (reject $r=1+\sqrt{2}$ $\because |r|<1$ as G.P is convergent)

First term of *H* = $(-2-4r)+4$

$=2-4r$

$=2-4(1-\sqrt{2})$

$=4\sqrt{2}-2$

5(a) GP is $1+\frac{3}{4}+\frac{9}{16}+\ldots$

Sum of first *n* terms = $S_n=\frac{1(1-0.75^n)}{1-0.75}$

Sum to infinity = $S_\infty=\frac{1}{1-0.75}$

$\frac{(1-0.75^n)}{1-0.75}>\frac{0.99}{1-0.75}$

$$0.75^n < 0.01$$

$$n > \frac{\ln 0.01}{\ln 0.75} = 16.01$$

least value of n is 17

(b) Area of the 10 sectors = Area of the circle

$$\frac{10}{2}(2a+9d) = \pi(2)^2$$

$$2a+9d = \frac{4\pi}{5} \quad \text{..... (1)}$$

$a,\ a+3d,\ a+8d, \ldots$ are in GP

$$\therefore \ \frac{a+3d}{a} = \frac{a+8d}{a+3d}$$

$$a^2+6ad+9d^2 = a^2+8ad$$

$$d = \frac{2a}{9}$$

Sub $d = \frac{2a}{9}$ into (1), $\therefore a = \frac{\pi}{5}$

Area of largest sector = *a* + 9*d* = *a* + 2*a*

$$= \frac{\pi}{5} + \frac{2\pi}{5} = \frac{3\pi}{5}$$

6) **(i)** $\frac{80}{2}\left[2(418.50)+79d\right] = 28740$

$$40\left[837+79d\right] = 28740$$

$$d = -1.5$$

$418.50 + 79(-1.5) = 300$ **($)**

(ii) $800r^{79} = 162.17$

$$r = \left(\frac{162.17}{800}\right)^{\frac{1}{79}}$$

Total cost $= \dfrac{800\left[1-\left(\dfrac{162.17}{800}\right)^{\frac{80}{79}}\right]}{1-\left(\dfrac{162.17}{800}\right)^{\frac{1}{79}}}$

≈ 32054.61

Extra cost $\approx 32054.61 - 28740$

$= 3314.61$ **($)**

(iii) $S_\infty = \dfrac{800}{1-\left(\dfrac{162.17}{800}\right)^{\frac{1}{79}}} = 40001.17$

7 **(i)** Start of 1st year, amount = \$80 x 50 = \$4000 (due to new members)

End of 1st year, amount = \$4000 (1.1) (due to interest from bank)

Start of 2nd year, amount = \$4000 (1.1) + \$4000

End of 2nd year, amount = \$4000 $(1.1)^2$ + \$4000 (1.1)

Start of 3rd year, amount = \$4000 $(1.1)^2$ + \$4000 (1.1) + \$4000

End of 3rd year, amount = \$4000 $(1.1)^3$ + \$4000 $(1.1)^2$ + \$4000 (1.1)

……………………………….

End of nth year, amount = \$4000 $(1.1)^n$ + \$4000 $(1.1)^{n-1}$ + … + \$4000 $(1.1)^2$ + \$4000 (1.1)

$= \dfrac{4000(1.1)(1.1^n - 1)}{1.1-1} = 44000(1.1^n - 1)$ (shown)

(ii) End of 25 years, number of members = $50 \times 25 = 1250$

Amt received per person = $44000(1.1^{25} - 1)/1250$
= \$346.18

8 (i)	$S_1 = P + \left(\frac{x}{100}\right)P = P\left(1+\frac{x}{100}\right)$ $S_2 = \left(P+S_1\right)\left(1+\frac{x}{100}\right) = \left[P + P\left(1+\frac{x}{100}\right)\right]\left(1+\frac{x}{100}\right)$ $= P\left[\left(1+\frac{x}{100}\right)+\left(1+\frac{x}{100}\right)^2\right]$
(ii)	$S_n = P\left[\left(1+\frac{x}{100}\right)+\left(1+\frac{x}{100}\right)^2+\ldots+\left(1+\frac{x}{100}\right)^n\right]$ $\left[\left(1+\frac{x}{100}\right)+\left(1+\frac{x}{100}\right)^2+\ldots+\left(1+\frac{x}{100}\right)^n\right]$ is a G..P. , with first term, $a = 1+\frac{x}{100}$ and common ratio, $r = 1+\frac{x}{100}$. $S_n = P\left(1+\frac{x}{100}\right)\frac{\left[\left(1+\frac{x}{100}\right)^n - 1\right]}{\left[\left(1+\frac{x}{100}\right)-1\right]}$ $= P\left(1+\frac{x}{100}\right)\left(\frac{100}{x}\right)\left[\left(1+\frac{x}{100}\right)^n - 1\right]$ $= P\left(\frac{100}{x}+1\right)\left[\left(1+\frac{x}{100}\right)^n - 1\right]$ (shown)
	Given $P = 1000,\ x = 4,\ S_n = 38000$ $1000\left(\frac{100}{4}+1\right)\left[\left(1+\frac{4}{100}\right)^n - 1\right] > 38000$ $\frac{104}{4}\left[\left(1.04\right)^n - 1\right] > 38$ $\left(1.04\right)^n > 2.46154$ $n\lg\left(1.04\right) > \lg\left(2.46154\right)$ $n > \frac{\lg\left(2.46154\right)}{\lg(1.04)}$ $n > 22.967$ Min. $n = 23$

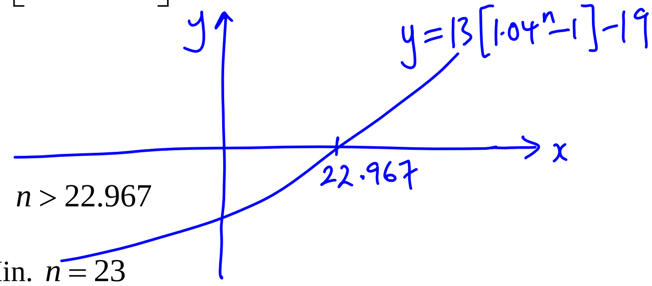
Alternative Method

Given $P = 1000,\ x = 4,\ S_n = 38000$

$$1000\left(\frac{100}{4}+1\right)\left[\left(1+\frac{4}{100}\right)^n - 1\right] > 38000$$

$$\frac{104}{4}\left[(1.04)^n - 1\right] > 38$$

$$13\left[(1.04)^n - 1\right] - 19 > 0$$



$\therefore\ n > 22.967$

Min. $n = 23$

9)

7 i) 3^{k-1}

ii) 3^{k-1}

iii) $3^{k-1} + 2(3^{k-1} - 1) = 3^k - 2$

Alternative:

Last integer in the k^{th} set
$= 1^{st}$ integer in the $(k+1)^{th}$ set $- 2 = 3^k - 2$

iv) $3^0 + 3^1 + 3^2 + \dots + 3^{k-1}$

$$= \frac{1(1-3^k)}{1-3}$$

$$= \frac{1}{2}(3^k - 1)$$

v) $1 + 3 + 5 + \dots + (3^k - 2)$

$$= \frac{\frac{1}{2}(3^k-1)}{2}(1 + 3^k - 2)$$

$$= \frac{1}{4}(3^k - 1)^2$$