AP/GP (Monetary Problems)

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5(i)	$u_0 = \$1.1^4(100\ 000) = \$146\ 410$
(ii)	$u_1 = 1.1(u_0 - 12x)$ $u_2 = 1.1[1.1(u_0 - 12x) - 12x]$ $= 1.1^2 u_0 - 1.1^2(12x) - 1.1(12x)$ $u_3 = 1.1[1.1^2 u_0 - 1.1^2(12x) - 1.1(12x) - 12x]$ $= 1.1^3 u_0 - 1.1^3(12x) - 1.1^2(12x) - 1.1(12x)$: : $u_n = 1.1^n u_0 - 1.1^n(12x) - 1.1^{n-1}(12x) - \ldots - 1.1(12x)$ $= 1.1^n u_0 - 12x\left(1.1^n + 1.1^{n-1} + \ldots + 1.1\right)$ $= 1.1^n u_0 - 12x\left(\frac{1.1\left(1.1^n - 1\right)}{0.1}\right)$ $= 1.1^n u_0 - 132x\left(1.1^n - 1\right)$
(iii)	n = 7 at end of 2020 $1.1^7 u_0 - 132x(1.1^7 - 1) < 1000$ $x > \$2270.30$ Least x to the nearest dollar = \$2271

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(a)	$ar = a + 3d \quad \Rightarrow \quad d = -\frac{1}{2}a$ Sum of first n even-numbered terms $= \frac{n}{2}\left[2\left(\frac{1}{2}a\right) + (n-1)\left(-a\right)\right]$ $= \frac{n}{2}[2a - an]$ $= \frac{an\left(2-n\right)}{2}$
(b)(i)	Amount owed at the beginning of the third month $= \left[\left(50000 - x\right)(1.035) - x\right](1.035)$ $= 50000(1.035^2) - (1.035 + 1.035^2)x$
(b)(ii)	Amount owed at the beginning of the nth month $= 50000(1.035^{n-1}) - (1.035 + 1.035^2 + \cdots + 1.035^{n-1})x$ For the repayment to be completed during the nth payment,

	$50000(1.035^{n-1})-(1.035+1.035^2+\cdots+1.035^{n-1})x\le x$ $50000(1.035^{n-1})\le(1+1.035+1.035^2+\cdots+1.035^{n-1})x$ $\le\frac{1.035^n-1}{1.035-1}x$ $\le\frac{1.035^n-1}{0.035}x$ Thus $x\ge\frac{1750(1.035^{n-1})}{1.035^n-1}$.

3(a)	Let the first term and common difference of the AP be a and d respectively. $\frac{a+2d}{a}=\frac{a+3d}{a+2d}$ $(a+2d)^2=a(a+3d)$ $a^2+4ad+4d^2=a^2+3ad$ $a=-4d$ ------ (1) $\frac{20}{2}\left[2(a+d)+(19)2d\right]=960$ $2a+40d=96$ ------ (2) Subs. (1) into (2): $-8d+40d=96$ $\Rightarrow d=3$
(b)(i)	(see table below)

Mth	Amt at beginning of mth	Amt at end of mth
Jan	x	$1.03x$
Feb	$1.03x + 2x$	$1.03^2x + 2(1.03)x$
Mar	$1.03^2x + 2(1.03)x + 3x$	$1.03^3x + 2(1.03)^2x + 3(1.03)x$

$\therefore$ Amt on the last day of March

= $\$[1.03^3x + 2(1.03)^2x + 3(1.03)x]$

= $\$x[1.03^3 + 1.03^2 + 1.03$
$+ 1.03^2 + 1.03$
$+ 1.03]$

$$=\$x\left[\frac{1.03(1.03^3-1)}{1.03-1}+\frac{1.03(1.03^2-1)}{1.03-1}+\frac{1.03(1.03^1-1)}{1.03-1}\right]$$

$$=\$x\,\frac{1.03(1.03^3+1.03^2+1.03-3)}{0.03}$$

	$= \$\frac{103x}{3}\left(\frac{1.03\left(1.03^3-1\right)}{0.03}-3\right)$ $= \$\frac{103x}{3}\left(\frac{103\left(1.03^3-1\right)-9}{3}\right)$ $= \$\ \frac{103x}{9}\left(103(1.03)^{12}-112\right)$ (shown)
(b) (ii)	$\frac{103x}{9}\left(103(1.03)^3-112\right)-\left(x+2x+3x\right)>100$ $\Rightarrow x>328.37$ $\therefore$ least value of x is 329

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(a) $S_n = S_{3n} - S_n \ \Rightarrow S_{3n} = 2S_n$

$\frac{3n}{2}\left[2a-\left(3n-1\right)\right]=n\left[2a-\left(n-1\right)\right]$

$a=\frac{7n-1}{2}$

(b)

*n*th birthday	Amount of Mrs Lee's contribution ($) on her daughter's *n*th birthday
0	1000
1	1000(1.05)+1000
2	(1000(1.05)+1000)(1.05) + 1000 = $1000\left(1.05\right)^2+1000\left(1.05\right)+1000$
3	$\left(1000\left(1.05\right)^2+1000\left(1.05\right)+1000\right)\left(1.05\right)$ $+1000$ $= 1000\left[\left(1.05\right)^3+\left(1.05\right)^2+\left(1.05\right)+1\right]$
…	…
17	$1000\left[\left(1.05\right)^{17}+\left(1.05\right)^{16}+\left(1.05\right)^{15}+\ldots+\left(1.05\right)^2+\left(1.05\right)+1\right]$
18	$1.05\times1000\left[\left(1.05\right)^{17}+\left(1.05\right)^{16}+\left(1.05\right)^{15}+\ldots+\left(1.05\right)^2+\left(1.05\right)+1\right]$

Mrs. Lee: $a=1000(1.05),\ r=1.05$

Total amount $= \$\frac{1050\left[\left(1.05\right)^{18}-1\right]}{1.05-1}$

= $29500 (3 s.f.)

*n*th birthday	Amount of Mr Lee's contribution ($) on his daughter's *n*th birthday
1	$\left[1.05(500)+1.05(500)\right]$
2	$1.05\left(2\times1.05(500)\right)+1.05^2(500)$ $= 3\times1.05^2(500)$
3	$1.05\left(3\times1.05^2(500)\right)+1.05^3(500)$ $= 4\times1.05^3(500)$
…	…
17	$18\times(1.05)^{17}(500)$
18	$\$1.05\left[18\times(1.05)^{17}(500)\right]$

Total amount = $\$\left[18\times(1.05)^{18}(500)\right]$ = $21700 (3 s.f.)

6 (a)(i)	Note that the progression of **A** is 1,3,5,7,9,11, … (consists of all positive odd numbers) *n*th term of **G**, $T_n = \frac{a}{2}\left(3^n - 1\right) - \frac{a}{2}\left(3^{n-1} - 1\right)$ $= \frac{a}{2}\left(3^n - 3^{n-1}\right) = \frac{a}{2}(3^{n-1})(3-1) = a(3^{n-1})$ 3^{n-1} is odd for all $n \geq 1$. Hence, for $a(3^{n-1})$ to be a term in **A**, then $a = 2m-1$, $m \in Z^+$. That is, *a* may be any positive odd integer.
(a)(ii)	Let the new progression be **H** : 2^{t_1}, 2^{t_2}, 2^{t_3}, 2^{t_4}, … Let *d* be the common difference of **A**. Consider, $\frac{(n+1)\text{th term of } \mathbf{H}}{(n)\text{th term of } \mathbf{H}}$ $= \frac{H_{n+1}}{H_n} = \frac{2^{t_{n+1}}}{2^{t_n}} = 2^{t_{n+1} - t_n} = 2^d = 2^2 = 4$, which is a constant. Hence **H** is a geometric progression.
(b)(i)	Value of the machine after 5 years $= 70000(0.9)^5$ = $41334.30.
(b)(ii)	Maximum revenue that could possibly be generated by the machine $= \frac{14000}{1-(0.93)} = \$200000$.

(b)(iii)	Consider $$\frac{14000(1-(0.93)^n)}{1-(0.93)}+70000(0.9)^n-70000\geq 40000$$ $$20(1-(0.93)^n)+7(0.9)^n-7\geq 4$$ $$20(0.93)^n-7(0.9)^n\leq 9$$ From G.C., n | a_n: … | …; 5 | 9.7803; 6 | 9.2197; 7 | 8.6859 $\therefore n\geq 7$ Hence, the number of years the machine is in operation is 7.

From G.C.,

n	a_n
…	…
5	9.7803
6	9.2197
7	8.6859