Applications of Differentiation Revision Exercise

Tangents and Normals

1. $y^2\sqrt{\left(1-4x^2\right)^3} = 9 + y\sin^{-1}(2x) \quad \Rightarrow \quad y^2\left(1-4x^2\right)^{3/2} = 9 + y\sin^{-1}(2x)$

When x = 0, $y^2(1) = 9 + y\sin^{-1}(0) = 9 \quad \Rightarrow \quad y = -3$ or $y = 3$

$$2y\frac{dy}{dx}\sqrt{\left(1-4x^2\right)^3} + y^2 \cdot \frac{3}{2}\left(1-4x^2\right)^{1/2} \cdot -8x = \frac{dy}{dx}\sin^{-1}(2x) + y \cdot \frac{2}{\sqrt{1-4x^2}}$$

At T (0, 3), $2(3)\frac{dy}{dx}(1) + 3^2(0) = \frac{dy}{dx}(0) + 3(2) \quad \Rightarrow \quad \frac{dy}{dx} = 1$

At N (0, –3), $2(-3)\frac{dy}{dx}(1) + (-3)^2(0) = \frac{dy}{dx}(0) + (-3)(2) \quad \Rightarrow \quad \frac{dy}{dx} = 1$

At T (0, 3), $y - 3 = x - 0 \quad \Rightarrow y = x + 3$
At N (0, –3), $y + 3 = -x - 0 \quad \Rightarrow y = -x - 3$

2(i)

$x = \cos^{-1}(2t), \qquad y = 4t^2$

$$\frac{dx}{dt} = \frac{-1}{\sqrt{1-(2t)^2}}(2) = \frac{-2}{\sqrt{1-(2t)^2}}, \qquad \frac{dy}{dt} = 8t$$

$$\therefore \frac{dy}{dx} = \frac{8t}{\frac{-2}{\sqrt{1-(2t)^2}}} = -4t\sqrt{1-4t^2}$$

At point P, where $t = \frac{1}{2\sqrt{2}}$,

$$\frac{dy}{dx} = -4\left(\frac{1}{2\sqrt{2}}\right)\sqrt{1-4\left(\frac{1}{2\sqrt{2}}\right)^2} = \frac{-2}{\sqrt{2}}\sqrt{1-4\left(\frac{1}{8}\right)} = \frac{-2}{\sqrt{2}}\sqrt{\frac{1}{2}} = -1$$

Equation of tangent at point P:

$$x = \cos^{-1}\left(2\left(\frac{1}{2\sqrt{2}}\right)\right) = \frac{\pi}{4}, \quad y = 4\left(\frac{1}{2\sqrt{2}}\right)^2 = \frac{1}{2}$$

$$y - \frac{1}{2} = -1\left(x - \frac{\pi}{4}\right) \Rightarrow y = -x + \frac{\pi}{4} + \frac{1}{2} \Rightarrow y = -x + \frac{\pi+2}{4}$$

(ii) At point A, where $y=0, x=\frac{\pi+2}{4}$ $\therefore A=\left(\frac{\pi+2}{4},0\right)$

At point B, where $x=0, y=\frac{\pi+2}{4}$ $\therefore B=\left(0,\frac{\pi+2}{4}\right)$

Area of $\Delta AOB=\frac{1}{2}\left(\frac{\pi+2}{4}\right)\left(\frac{\pi+2}{4}\right)=\frac{(\pi+2)^2}{32}$

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$x=2u^2 \quad , \quad y=\frac{2}{u}$

$\frac{dx}{du}=4u \quad , \quad \frac{dy}{du}=\frac{-2}{u^2}$

(i)

$\therefore \frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}=\frac{-2}{u^2}\times\frac{1}{4u}=-\frac{1}{2u^3}.$

At $u=1, \frac{dy}{dx}=-\frac{1}{2}, x=2, y=2.$

Eqn of tangent: $y-2=-\frac{1}{2}(x-2)$

$$y=-\frac{1}{2}x+3$$

Alternative : $u=\frac{2}{y}$

$$x=2\left(\frac{2}{y}\right)^2$$

$$y^2=\frac{8}{x}$$

Diff wrt x , $2y\frac{dy}{dx}=-\left(\frac{8}{x^2}\right)$

At $u=1, \frac{dy}{dx}=-\frac{1}{2}, x=2, y=2.$

Eqn of tangent: $y-2=-\frac{1}{2}(x-2)$

$$y=-\frac{1}{2}x+3$$

At pts of intersection between the tangent and the curve,

$$\frac{2}{u} = -\frac{1}{2}\left(2u^2\right)+3$$

$$4 = -2u^3 + 6u$$

(ii) $u^3 - 3u + 2 = 0$

$$(u-1)(u^2+u-2) = 0$$

$$(u-1)(u+2)(u-1) = 0$$

$u = 1$ or $u = -2$

$\therefore$ the tangent will intersect the curve again.

Alternative : Solve $y^2 = \frac{8}{x}$ and $y = -\frac{1}{2}x + 3$ simultaneously

Obtain $y^3 - 3y^2 + 4 = 0$

y = 2, 2, −1 and u = 1 , −2

$\therefore$ the tangent will intersect the curve again.

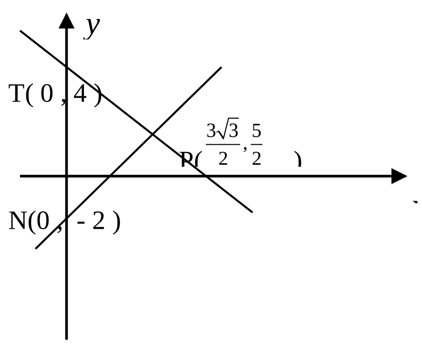
gradient of normal $= 2u^3$

(iii) $2u^3 = \frac{1}{4}$

$$u = \frac{1}{2}$$

4.

12 (a)	$x = 3\cos\theta \qquad y = 2+\sin\theta$ $\frac{dy}{d\theta} = -3\sin\theta \qquad \frac{dy}{d\theta} = \cos\theta$ $\frac{dy}{dx} = \frac{\cos\theta}{-3\sin\theta} = -\frac{1}{3}\cot\theta$ Where $\theta = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{1}{3}\cot\left(\frac{\pi}{6}\right) = -\frac{1}{3}\frac{1}{\tan\left(\frac{\pi}{6}\right)} = -\frac{\sqrt{3}}{3}$ Point P is $\left(3\cos\frac{\pi}{6}, 2+\sin\frac{\pi}{6}\right) = \left(\frac{3\sqrt{3}}{2}, \frac{5}{2}\right)$. Equation of tangent is	[B1] [M1]

	$y-\frac{5}{2}=-\frac{\sqrt{3}}{3}\left(x-\frac{3\sqrt{3}}{2}\right)$ $y+\frac{\sqrt{3}}{3}x-4=0$ (Shown)	[A1] AG
(b)	Gradient of the normal: $-\frac{1}{-\frac{\sqrt{3}}{3}}=\frac{3}{\sqrt{3}}=\sqrt{3}$ Equation of normal is $y-\frac{5}{2}=\sqrt{3}\left(x-\frac{3\sqrt{3}}{2}\right)$ $y=\sqrt{3}x-2$	[B1] [M1] [A1]
(c)	Eqn of tangent : $y+\frac{\sqrt{3}}{3}x-4=0$ When x = 0, y = 4. T is $(0,4)$ Equation of normal : $y=\sqrt{3}x-2$ When x = 0, y = -2. N is (0, -2). Area of triangle $=\frac{1}{2}\times NT\times\frac{3\sqrt{3}}{2}$ $=\frac{1}{2}\times(4-(-2))\times\frac{3\sqrt{3}}{2}$ $=\frac{9\sqrt{3}}{2}$ units^2  	[B1] ½ each [M1] [A1] exact ans
(d)	Let $u=xy=(3\cos\theta)(2+\sin\theta)$ $=6\cos\theta+3\sin\theta\cos\theta$ $=6\cos\theta+\frac{3}{2}\sin 2\theta$ $\therefore \quad \frac{du}{d\theta}=-6\sin\theta+3\cos 2\theta$ $\frac{d\theta}{dt}=\left(\frac{d\theta}{dx}\right)\left(\frac{dx}{dt}\right)=-\frac{1}{3\sin\theta}\left(\frac{1}{10}\right)=-\frac{1}{30\sin\theta}$ $\frac{du}{dt}=\left(\frac{du}{d\theta}\right)\left(\frac{d\theta}{dt}\right)$	 [B1] [M1]

	$= \left(-6\sin\theta + 3\cos 2\theta\right)\left(-\dfrac{1}{30\sin\theta}\right)$	[A1]
	When $\theta = \dfrac{\pi}{6}$,	
	$\dfrac{du}{dt} = \left(\dfrac{du}{d\theta}\right)\left(\dfrac{d\theta}{dt}\right)$	
	$= \left(-6\sin\dfrac{\pi}{6} + 3\cos\dfrac{\pi}{3}\right)\left(-\dfrac{1}{30\sin\dfrac{\pi}{6}}\right)$	[M1]
	$= 0.1$ sq. units / s	[A1]

Graphical Interpretations

5.

Q6 (i) $x = e^{2t} + \cos t \Rightarrow \frac{dx}{dt} = 2e^{2t} - \sin t$

$y = e^{2t} + \sin t \Rightarrow \frac{dy}{dt} = 2e^{2t} + \cos t$.

$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{2e^{2t} + \cos t}{2e^{2t} - \sin t}$.

For $0 < t \leq \frac{\pi}{2}$, $\cos t \geq 0$, $2e^{2t} > 0$ $\therefore 2e^{2t} + \cos t > 0$ i.e. $2e^{2t} + \cos t \neq 0$

Thus, $\frac{dy}{dx} \neq 0$, i.e. no stationary points

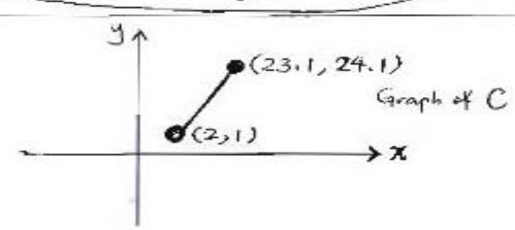
Alternatively, suppose $\frac{dy}{dx} = 0$. Then $2e^{2t} + \cos t = 0$

i.e. $2e^{2t} = -\cos t$

There is no point of intersection between the graphs of $y = 2e^{2x}$ and $y = -\cos x$, as seen from GC, when $0 < x \leq \frac{\pi}{2}$.

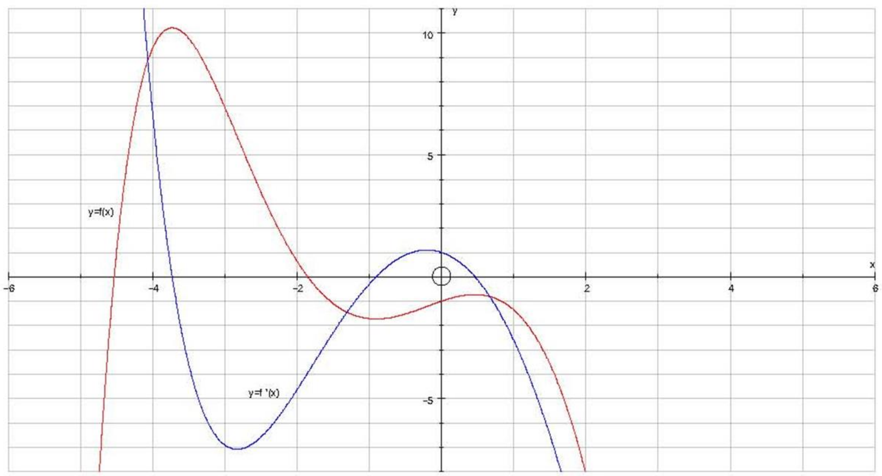
$\therefore$ no stationary points.

(ii)



6.

(a) $y=-x^3-e^{-x} \Rightarrow \dfrac{dy}{dx}=-3x^2+e^{-x}$



The 3 x - intercepts of the graph $y=\mathrm{f}\,'(x)$ represents the x-coordinates of the 3 stationary points of the graph $y=\mathrm{f}(x)$.

7.

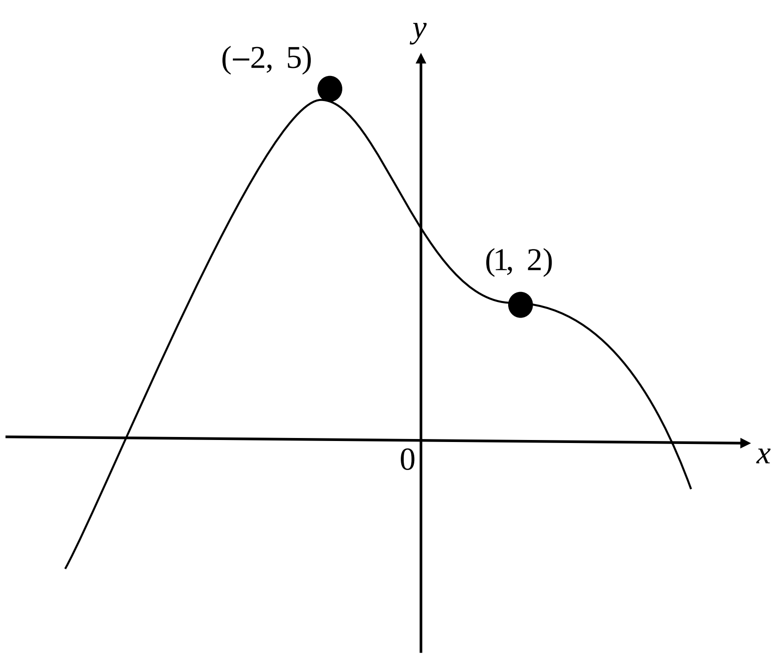
(a) At stationary points, $\mathrm{f}\,'(x)=0 \;\Rightarrow\; x=-2,\ 1$

Hence the coordinates of the stationary points are $\underline{\underline{(-2,5) \text{ and } (1,\ 2)}}$ **[A1]**

$\mathrm{f}\,'((-2)^-)>0$ and $\mathrm{f}\,'((-2)^+)<0$,
by First Derivative Test, $(-2,5)$ is a <u>maximum</u> point.
$\mathrm{f}\,'(1^-)<0$ and $\mathrm{f}\,'(1^+)<0$,
by First Derivative Test, $(1,\ 2)$ is a stationary point of <u>inflexion</u>. **[A1]**

(b) $y=\mathrm{f}(x)$ is decreasing when $\mathrm{f}\,'(x)<0$, i.e. when $\underline{\underline{-2<x<1 \text{ or } x>1}}$. **[A1]**

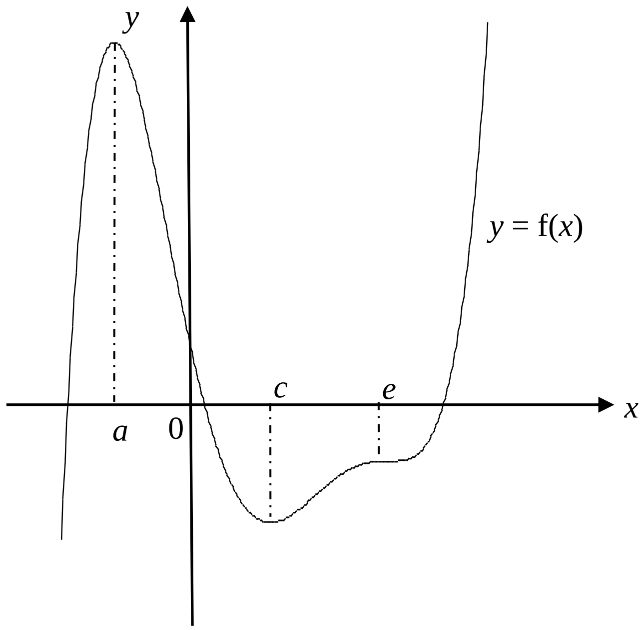
(c) Graph of $y=\mathrm{f}(x)$



8 (i)

x-coordinate of stationary point	Nature
a	Maximum
c	Minimum
e	Stationary point of inflexion

(ii)



Maxima and Minima

10.

$2y + x = 12 \Rightarrow y = 6 - \dfrac{x}{2}$

$h = \sqrt{y^2 - \left(\tfrac{1}{2}x\right)^2}$

$= \sqrt{\left(6 - \tfrac{1}{2}x\right)^2 - \left(\tfrac{1}{2}x\right)^2}$

$= \sqrt{\left(6 - \tfrac{1}{2}x - \tfrac{1}{2}x\right)\left(6 - \tfrac{1}{2}x + \tfrac{1}{2}x\right)}$

$= \sqrt{(6 - x)6}$

$= \sqrt{36 - 6x}$

Area of triangle, $A = \dfrac{1}{2} \times \text{base} \times \text{height}$

$= \dfrac{x}{2}\sqrt{36 - 6x}$

$$\frac{dA}{dx} = \frac{1}{2}\sqrt{36-6x} + \frac{x}{2}\left(\frac{-6}{2\sqrt{36-6x}}\right)$$

$$= \frac{36-6x-3x}{2\sqrt{36-6x}}$$

$$= \frac{36-9x}{2\sqrt{36-6x}}$$

When $\frac{dA}{dx} = 0$

$$36-9x = 0$$

$$x = 4$$

x	4^-	4	4^+
$\frac{dA}{dx}$	+ /	0 —	- \

Alternative solution:

$\frac{d^2A}{dx^2} = -1.30\ (3\ sig.\ fig),$

Hence area is max. when x = 4.

Since $2y + x = 12 \Rightarrow y = 4$

$\therefore$ Area is a maximum when triangle is equilateral.

11. Objective : Form equation in 2 variable quantities and solve a minimum quantity problem.

(i) $\pi r^2 h = V$

$$h = \frac{V}{\pi r^2}$$

$$A = 2\pi r^2 + 2\pi r h$$

$$= 2\pi r^2 + 2\pi r\left(\frac{V}{\pi r^2}\right)$$

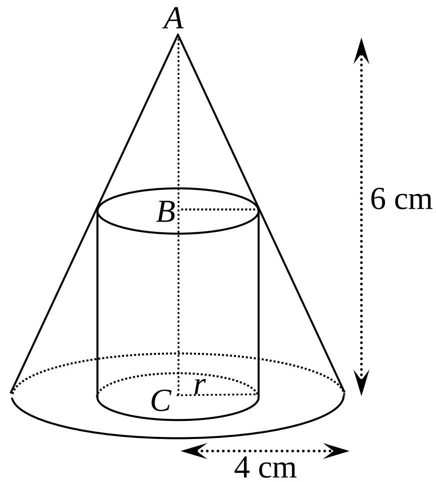
$$= 2\pi r^2 + \left(\frac{2V}{r}\right)$$

(ii) For min A, $\frac{\mathrm{d}A}{\mathrm{d}r} = 4\pi r - \frac{2V}{r^2} = 0$

$$4\pi r^3 = 2V$$

$$r = \left(\frac{V}{2\pi}\right)^{\frac{1}{3}}$$

12.



Solution

3i) AB = 6 – h $\Rightarrow$ $\dfrac{AB}{r} = \dfrac{6}{4}$ $\Rightarrow$ $\dfrac{6-h}{r} = \dfrac{3}{2}$ $\Rightarrow$ h = 6 – $\dfrac{3}{2}r$

ii) V = $\pi r^2 h = \pi r^2 \left(6 - \dfrac{3}{2}r\right) = \pi\left(6r^2 - \dfrac{3}{2}r^3\right)$ $\Rightarrow$ $\dfrac{dV}{dr} = \pi\left(12r - \dfrac{9r^2}{2}\right) = 0$

$\Rightarrow$ $r = 0$ (NA) or $r = \dfrac{8}{3}$

r	$\left(\frac{8}{3}\right)^-$	$\frac{8}{3}$	$\left(\frac{8}{3}\right)^+$	
$\frac{dV}{dr}$	+ve	0	–ve	$\therefore$ max. V at $r = \frac{8}{3}$
Slope	/	—	\	

Max. V = $128\pi/9$ cm^3 or 14.2π cm^3

3

Alternative test:

$\dfrac{d^2V}{dr^2} = \pi(12 - 9r) = -12\pi < 0$ when $r = \dfrac{8}{3}$.

Alternatively, may write V in terms of h:

	$V = \pi\left(4-\frac{2}{3}h\right)^2 h = \pi\left(16h-\frac{16}{3}h^2+\frac{4}{9}h^3\right)$ $\frac{dV}{dh} = \pi\left(16-\frac{32}{3}h+\frac{4h^2}{3}\right) = \frac{4}{3}\pi(h-6)(h-2) = 0$ $h = 2$ cm (as $h \neq 6$ cm)

Rate of Change

13.

$x = 2\tan 2\theta \quad \Rightarrow \quad dx/d\theta = 4\sec^2 2\theta$

$y = \ln(2 + e^{\theta^2-\theta}) \quad \Rightarrow \quad dy/d\theta = \frac{(2\theta-1)e^{\theta^2-\theta}}{2+e^{\theta^2-\theta}}$

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{(2\theta-1)e^{\theta^2-\theta}}{2+e^{\theta^2-\theta}}\cdot\frac{1}{4\sec^2 2\theta}$$

When $\theta = 0$, $\frac{dy}{dx} = \frac{-1}{2+1}.\frac{1}{4} = -\frac{1}{12} \quad \Rightarrow \quad \frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt} = -\frac{1}{12}\times 0.6 = -0.05$ units s^{-1}

Alternative,

$x = 2\tan 2\theta \quad \Rightarrow \quad dx/d\theta = 4\sec^2 2\theta$

$y = \ln(2 + e^{\theta^2-\theta}) \quad \Rightarrow \quad dy/d\theta = \frac{(2\theta-1)e^{\theta^2-\theta}}{2+e^{\theta^2-\theta}}$

$$\frac{dy}{dt} = \frac{dy}{d\theta}.\frac{d\theta}{dx}.\frac{dx}{dt} = \frac{(2\theta-1)e^{\theta^2-\theta}}{2+e^{\theta^2-\theta}}\cdot\frac{1}{4\sec^2 2\theta}.(0.6)$$

$$= \frac{(-1)e^0}{2+e^0}\cdot\frac{1}{4\sec^2 0}.(0.6) = -(0.1)/2 = -0.05$$

14.

(b) $\frac{dy}{dx} = -\sin x e^{\cos x} + e^{\frac{1}{2}} 2\sin x \cos x$

$or \quad = -\sin x e^{\cos x} + e^{\frac{1}{2}} \sin 2x$

$\frac{dx}{dt} = -4;$

$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt}$

$\frac{dy}{dt} = -4\left(-\sin x e^{\cos x} + e^{\frac{1}{2}} \sin 2x\right)$

$or \quad 4\sin x e^{\cos x} - 4e^{\frac{1}{2}} \sin 2x$

When $x = \frac{\pi}{2}$,

$\frac{dy}{dt} = -4\left(-\sin\frac{\pi}{2} e^{\cos\frac{\pi}{2}} + e^{\frac{1}{2}} \sin 2(\frac{\pi}{2})\right) = 4$

15.

3(i)	Let $a = \pi r^2$ be the area of the smaller circle. Then $\pi R^2 - a = 25 \Rightarrow a = \pi R^2 - 25$. $\Rightarrow \frac{\mathrm{d}a}{\mathrm{d}t} = 2\pi R \frac{\mathrm{d}R}{\mathrm{d}t} = 2\pi R$. (since $\frac{\mathrm{d}R}{\mathrm{d}t} = 1$) When $R = 5$, $\frac{\mathrm{d}a}{\mathrm{d}t} = 2\pi(5) = 10\pi$. Hence, required rate = 10π cm^2/min . □
(ii)	$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}a}\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{2\pi r}(10\pi) = \frac{5}{r}$. Now, $a = \pi r^2 = \pi R^2 - 25 \Rightarrow r = \sqrt{\frac{\pi R^2 - 25}{\pi}}$. (since $r > 0$) Hence, $\frac{\mathrm{d}r}{\mathrm{d}t} = 5\sqrt{\frac{\pi}{\pi R^2 - 25}}$. When $R = 5$, $\frac{\mathrm{d}r}{\mathrm{d}t} = 5\sqrt{\frac{\pi}{25\pi - 25}} = \sqrt{\frac{\pi}{\pi - 1}}$. Hence, required rate = $\sqrt{\frac{\pi}{\pi - 1}}$ cm/min . □

16.

Objective : Use of basic trigonometry to form equation with one variable , use of chain rule

$$A = 2\pi rl = 2\pi\left(h\tan 30^\circ\right)\left(\frac{h}{\cos 30^\circ}\right)$$

$$= 2\pi h\left(\frac{1}{\sqrt{3}}\right)\left(h \div \frac{\sqrt{3}}{2}\right) = \frac{4}{3}\pi h^2 \quad \text{(shown)}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}h}\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{8}{3}\pi h\frac{\mathrm{d}h}{\mathrm{d}t}$$

$h = 8,$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}h}\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{8}{3}\pi\left(8\right)\left(-0.36\right) = -7.68\pi\ cm^2 s^{-1}$$