Revision Exercise on Applications of Integration

1 Solution

Objectives : Find area bounded by 2 curves and finding volume of solid of revolution by using the difference between the volume of 2 solids of revolution

(a) Area of $R = \int_1^2 (x+1)\ln x \, \mathrm{d}x - \frac{1}{2}(1)(2)$

$= \left[\left(\frac{1}{2}x^2 + x\right)\ln x\right]_1^2 - \int_1^2 \left(\frac{1}{2}x^2 + x\right)\left(\frac{1}{x}\right)\mathrm{d}x - 1$

$= 4\ln 2 - \int_1^2 \left(\frac{1}{2}x + 1\right)\mathrm{d}x - 1$

$= 4\ln 2 - \left[\frac{1}{4}x^2 + x\right]_1^2 - 1$

$= 4\ln 2 - \frac{11}{4}$

(b) Volume $= \pi\int_1^2 (x+1)^2(\ln x)^2 \, \mathrm{d}x - \frac{1}{3}\pi(2)^2(1) = 0.225$

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2 Solution

8a) Area of region R = $\int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right) \mathrm{d}x$

$$= \left[\tan^{-1} x - \frac{1}{2}\ln(1+x^2)\right]_0^1$$

$$= \tan^{-1} 1 - \frac{1}{2}\ln(2)$$

$$= \frac{\pi}{4} - \frac{1}{2}\ln(2)$$

$\therefore k = \frac{1}{2}$

8b) (i) Coordinates of A = (1.9, 0.9)

(ii) $\int \sin^2 x \quad \mathrm{d}x = \int \left(\frac{1-\cos 2x}{2}\right) \quad \mathrm{d}x$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + C$$

(iii) Volume generated about x-axis

$$= \pi \int_0^{1.9} \sin^2 x \quad \mathrm{d}x - \frac{1}{3}\pi(0.9)^2(1.9)$$

$$= \pi \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{1.9} - \frac{1}{3}\pi(0.9)^2(1.9)$$

$$= \pi \left[\frac{1.9}{2} - \frac{1}{4}\sin 3.8\right] - \frac{1}{3}\pi(0.9)^2(1.9)$$

$$= 1.9$$

Or using GC, answer is 1.7

Or: $\pi \int_0^{1.9} \sin^2 x - \left(\frac{1}{2}x\right)^2 \quad \mathrm{d}x$ to get 1.7

4 **Solution**

$y = x - 4 + \frac{1}{x-2} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{(x-2)^2}$

At $x = 4$, $\frac{dy}{dx} = \frac{3}{4}$.

Gradient of normal = $-\frac{4}{3}$.

$y - \frac{1}{2} = -\frac{4}{3}(x-4)$

$y = -\frac{4}{3}x + \frac{35}{6}$ (Shown)

Let Q be the other point of intersection.
By GC, $Q = (2.2143, 2.8809)$

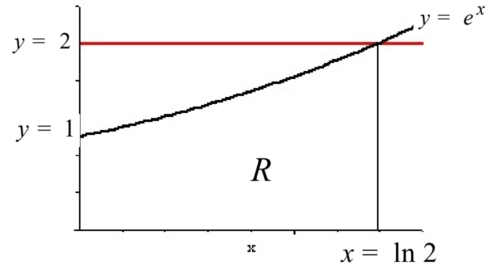
Volume generated by R

$$= \pi \int_{2.2143}^{4} \left(-\frac{4}{3}x + \frac{35}{6}\right)^2 - \left(x - 4 + \frac{1}{x-2}\right)^2 \, dx$$

$= 16.477$

[6]

5 Solution

(a)	Area= $\int_0^{\ln 2} y\,dx = \int_0^{\ln 2} e^x\,dx = \left[e^x\right]_0^{\ln 2}$	**B1**- Correct expression $\int_0^{\ln 2} e^x\,dx$
	$= e^{\ln 2} - e^0 = 1$	**A1**- Correct answer
	$\int_1^2 \ln y\,dy = \int_1^2 x\,dy =$ Rectangle Area - Region R $= 2\ln 2 - 1$ (exact)	**M1**- Equate $\int_1^2 \ln y\,dy$ to difference of rectangle and R **A1**- Correct **exact** answer
		
(b)	<u>Volume about y-axis</u> $V_1 = \pi\int_1^2 \left(\frac{2}{y}\right)^2 dy = -\pi\left[\frac{4}{y}\right]_1^2 = 2\pi$	**M1** – Express volume ($from\ y = \frac{2}{x}, y = 1, y = 2\ \&\ y-axis$) as $\pi\int\left(\frac{2}{y}\right)^2 dy$ (ignore limits) **A1** - Obtain answer
	$V_2 = \pi\int_1^2 (y-1)^2 dy = \pi\left[\frac{(y-1)^3}{3}\right]_1^2 = \frac{\pi}{3}$	**M1-** Express volume ($from\ y = x+1, y = 2\ \&\ y-axis$) as $\frac{1}{3}\pi r^2 h$ or $\pi\int (y-1)^2 dy$ (ignore values of *r*, *h* & limits)
	Volume = $V_1 - V_2 = \frac{5\pi}{3}$	**A1**- Obtain correct final answer

6 Solution

$$x=\sqrt{12}\cos\theta \Rightarrow \frac{dx}{d\theta}=-\sqrt{12}\sin\theta$$

$$x=\sqrt{3}, \sqrt{12}\cos\theta=\sqrt{3} \Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$$

$$x=0, \sqrt{12}\cos\theta=0 \Rightarrow \theta=\frac{\pi}{2}$$

$$\int_0^{\sqrt{3}} \sqrt{12-x^2}\,dx=\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{12-12\cos^2\theta}\left(-\sqrt{12}\sin\theta\right)\,d\theta$$

$$=\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 12\sin^2\theta\;d\theta$$

$$=6\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(1-\cos 2\theta\right)\,d\theta$$

$$=6\left[\theta-\frac{1}{2}\sin 2\theta\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}$$

$$=6\left[\frac{\pi}{2}-\frac{1}{2}\sin\pi-\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}\right]$$

$$=\pi+\frac{3\sqrt{3}}{2}$$

(i) **Intersection points:** Substitute $y=x^2$ into $x^2+y^2=12$

$$y^2+y=12 \Rightarrow (y+4)(y-3)=0 \Rightarrow y=-4 \text{ (rejected)}, 3$$

$$x^2=3 \Rightarrow x=\sqrt{3}, -\sqrt{3}$$

$$A=2\left(\int_0^{\sqrt{3}} \sqrt{12-x^2}\,dx-\int_0^{\sqrt{3}} x^2\,dx\right)$$

$$=2\pi+3\sqrt{3}-2\left[\frac{x^3}{3}\right]_0^{\sqrt{3}}$$

$$=2\pi+\sqrt{3}$$

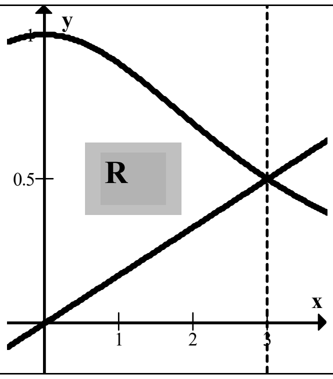
(ii) Required volume

$$=\int_0^3 \pi y\;dy+\int_3^{\sqrt{12}} \pi\left(12-y^2\right)\,dy$$

$$=\pi\left[\frac{y^2}{2}\right]_0^3+\pi\left[12y-\frac{y^3}{3}\right]_3^{\sqrt{12}}$$

$$=\pi\left(16\sqrt{3}-\frac{45}{2}\right)$$

7 Solution

11. (i)	$x = 3\tan\theta$ $\frac{dx}{d\theta} = 3\sec^2\theta$
	When $x = 0$, $\theta = 0$. When $x = 3$, $\theta = \frac{\pi}{4}$.
	$\int_0^3 y^2 dx = \int_0^3 \left(\frac{9}{x^2+9}\right)^2 dx$ $= \int_0^{\frac{\pi}{4}} \left(\frac{9}{(3\tan\theta)^2+9}\right)^2 \left(3\sec^2\theta d\theta\right)$
	$= \int_0^{\frac{\pi}{4}} \left(\frac{9}{9\left(\tan^2\theta+1\right)}\right)^2 \left(3\sec^2\theta d\theta\right) = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\sec^2\theta}\right)^2 \left(3\sec^2\theta d\theta\right)$ $= 3\int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta} d\theta = 3\int_0^{\frac{\pi}{4}} \cos^2\theta d\theta$
(ii)	
	Required Volume $= \pi\int_0^3 y^2 dx - \pi\int_0^3 \left(\frac{x}{6}\right)^2 dx$ $= 3\pi\int_0^{\frac{\pi}{4}} \cos^2\theta d\theta$ – Volume of cone $= 3\pi\int_0^{\frac{\pi}{4}} \frac{\cos 2\theta + 1}{2} d\theta - \frac{1}{3}\pi\left(\frac{1}{2}\right)^2 (3)$ $= 3\pi\left[\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_0^{\frac{\pi}{4}} - \frac{1}{4}\pi$
	$= 3\pi\left[\frac{1}{4} + \frac{\pi}{8}\right]_0^{\frac{\pi}{4}} - \frac{1}{4}\pi$ $= \frac{1}{2}\pi + \frac{3}{8}\pi^2$
	$\therefore a = \frac{1}{2}, b = \frac{3}{8}$

8 Solution

9(i)	Coordinates of the points of intersection are (–3, 3) and (2, –2).
(ii)	**Method 1:** Area of shaded region $= \int_{-2}^{3} \left(6 - y^2\right) \mathrm{d}y - \int_{-2}^{3} \left(-y\right) \mathrm{d}y$ $= \frac{125}{6}$ units^2 or 20.8 units^2 (to 3 s.f.) **Method 2:** Area of required region $= \left(\int_{-3}^{2} \sqrt{6-x}\, \mathrm{d}x - \frac{1}{2}(3)(3)\right) + \left(2\int_{2}^{6} \sqrt{6-x}\, \mathrm{d}x + \frac{1}{2}(2)(2)\right)$ (Area of region above x - axis + Area of region below x - axis) $= \frac{125}{6}$ units^2 or 20.8 units^2 (to 3 s.f.) **Method 3:** Area of required region $= \int_{-3}^{2} \left(\sqrt{6-x} - (-x)\right) \mathrm{d}x + 2\int_{2}^{6} \sqrt{6-x}\, \mathrm{d}x$ (Area of region between curves from $x = -3$ to $x = 2$) + (Area of region bounded by curve $y^2 = 6 - x$ and line $x = 2$) $= \frac{125}{6}$ units^2 or 20.8 units^2 (to 3 s.f.)
(ii)	Volume of R $= \pi\int_{-\sqrt{6}}^{-2} \left(6 - y^2\right)^2 \mathrm{d}y + \frac{1}{3}\pi(2)^2(2)$ **OR** $\pi\int_{-\sqrt{6}}^{-2} (6-y^2)^2\, \mathrm{d}y + \pi\int_{-2}^{0} (-y)^2\, \mathrm{d}y$ $= 10.36$ units^3 (to 2 d.p.)

9 Solution

(i).

$$\int e^{-t}\cos nt\ \mathrm{d}t$$

$$=\left[-e^{-t}\cos nt\right]-\int e^{-t}.\left(n\sin nt\right)\mathrm{d}t$$

$$=\left[-e^{-t}\cos nt\right]-\left\{\left[-ne^{-t}\sin nt\right]-\int -e^{-t}.\left(n^2\cos nt\right)\mathrm{d}t\right\}$$

Rearranging,

$$\left(1+n^2\right)\int e^{-t}\cos nt\ \mathrm{d}t=e^{-t}\left[n\sin nt-\cos nt\right]$$

$$\int e^{-t}\cos nt\ \mathrm{d}t=\frac{e^{-t}}{1+n^2}\left(n\sin nt-\cos nt\right)+B\ \text{(shown)}$$

(ii).

When $y=0$, $\sin t=0\Rightarrow\therefore t=\pi\ \left(\because\frac{\pi}{2}\le t\le\pi\right)$

$$\Rightarrow\therefore b=e^{-\pi}\cos 2\pi=e^{-\pi}$$

When $x=0$, $\cos 2t=0\Rightarrow\therefore t=\frac{3\pi}{4}\quad\left(\because\frac{\pi}{2}\le t\le\pi\right)$

$$\Rightarrow\therefore a=\sin\frac{3\pi}{4}=\frac{1}{\sqrt{2}}$$

$$\int_0^a x\ \mathrm{d}y=\int_0^{\frac{1}{\sqrt{2}}}x\ \mathrm{d}y=\int_\pi^{\frac{3\pi}{4}}\left(e^{-t}\cos 2t\right)\left(\cos t\right)\mathrm{d}t$$

$$=\frac{1}{2}\int_\pi^{\frac{3\pi}{4}}e^{-t}\left(2\cos 2t\cos t\right)\mathrm{d}t$$

$$=\frac{1}{2}\int_\pi^{\frac{3\pi}{4}}e^{-t}\left(\cos 3t+\cos t\right)\mathrm{d}t$$

$$=\frac{1}{2}\int_\pi^{\frac{3\pi}{4}}e^{-t}\cos 3t\ \mathrm{d}t+\frac{1}{2}\int_\pi^{\frac{3\pi}{4}}e^{-t}\cos t\ \mathrm{d}t$$

$$=\frac{1}{2}\left[\frac{e^{-t}}{10}\left(3\sin 3t-\cos 3t\right)\right]_\pi^{\frac{3\pi}{4}}+\frac{1}{2}\left[\frac{e^{-t}}{2}\left(\sin t-\cos t\right)\right]_\pi^{\frac{3\pi}{4}}$$

$$=\frac{1}{2}\left[\frac{e^{-\frac{3\pi}{4}}}{10}\left(\frac{3}{\sqrt{2}}-\frac{1}{\sqrt{2}}\right)-\frac{e^{-\pi}}{10}(0+1)\right]+\frac{1}{2}\left[\frac{e^{-\frac{3\pi}{4}}}{2}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)-\frac{e^{-\pi}}{2}(0+1)\right]$$

$$=\frac{1}{2}\left[\frac{e^{-\frac{3\pi}{4}}}{10}\left(\frac{2}{\sqrt{2}}\right)-\frac{e^{-\pi}}{10}\right]+\frac{1}{2}\left[\frac{e^{-\frac{3\pi}{4}}}{\sqrt{2}}-\frac{e^{-\pi}}{2}\right]$$

$$=\frac{3}{10}\left[\left(\sqrt{2}\right)e^{-\frac{3\pi}{4}}-e^{-\pi}\right]\ \text{(shown)}$$

10 Solution

(i) Sum of areas $= \frac{1}{n}\left[\frac{1}{\sqrt{4-\left(\frac{1}{n}\right)^2}}+\frac{1}{\sqrt{4-\left(\frac{2}{n}\right)^2}}+\ldots+\frac{1}{\sqrt{4-\left(\frac{n-1}{n}\right)^2}}+\frac{1}{\sqrt{4-\left(\frac{n}{n}\right)^2}}\right]$ **[M1]**

$$=\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-2^2}}+\ldots+\frac{1}{\sqrt{4n^2-(n-1)^2}}+\frac{1}{\sqrt{4n^2-n^2}}\right]$$ **[A1]**

$$=\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}}$$

(ii) Sum of areas of rectangles $\geq$ Actual area under the curve **[M1]**

$$\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}} \geq \int_0^1 \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x$$

$$\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}} \geq \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$$ **[A1]**

$$\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}} \geq \frac{\pi}{6}$$

(iii) Since $\sum_{r=1}^{100}\frac{1}{\sqrt{4n^2-r^2}} \approx \frac{\pi}{6}$

hence, $\pi \approx 6(0.52399) = 3.1439$ **[A1]**
