Complex Numbers

1. $(-2+3i)^2+\lambda(-2+3i)+\mu=0$

$(-2\lambda+\mu)+3\lambda i=5+12i$

$-2\lambda+\mu=5 \quad \& \quad 3\lambda=12$

Solving: $\lambda=4 \quad \& \quad \mu=13$

2. $z = w + 3i + 2$ ......(1)

$z^2-iw+5-2i=0$ ....(2)

(1)x i: $iz=iw+(-3)+2i$ ....(3)

(2)+(3): $iz=z^2+2$

$z^2-iz+2=0$

Solving using formula: $z=-i,\ 2i$

When $z=-i,\ w=-2-4i$

When $z=2i,\ w=-2-i$

3. Given $-1 + 2i$ is a root of the equation,

$\Rightarrow (-1+2i)^3+3(-1+2i)^2+a(-1+2i)+b=0$

$\Rightarrow a(-1+2i)+b=-2+14i$

$\Rightarrow (-a+b)+2ai=-2+14i$

$\Rightarrow -a+b=-2 \quad \& \quad 2a=14$

$\Rightarrow a=7 \quad \& \quad b=5$

$\therefore z^3+3z^2+7z+5=0$

Using GC, $z = -1,\ -1+2i,\ -1-2i$

Given $-1 + 2i$ is a root of the equation, the other root is $-1 - 2i$.

$z^3+3z^2+7z+5=(z-c)(z-(-1+2i))(z-(-1-2i))$

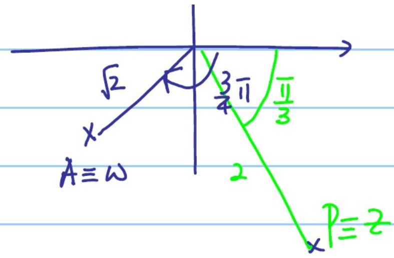
$=(z-c)(z^2+2z+5)$

$\Rightarrow c=1$

$\therefore$ the roots are $-1,\ -1+2i,\ -1-2i$

$|w| = \sqrt{2}$, $\arg w = -\frac{3}{4}\pi$

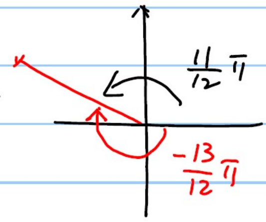
$|z| = 2$, $\arg z = -\frac{1}{3}\pi$



$|wz| = |w|\,|z| = \sqrt{2} \cdot 2 = 2\sqrt{2}$

$\arg(wz) = \arg(w) + \arg(z)$

$\arg(w) + \arg(z) = \left(-\frac{3}{4}\pi\right) + \left(-\frac{1}{3}\pi\right) = \frac{-9-4}{12}\pi = \frac{-13}{12}\pi$



$\therefore \arg(wz) = \frac{11}{12}\pi$

$w = \sqrt{2}\left[\cos\left(-\frac{3}{4}\pi\right) + i\sin\left(-\frac{3}{4}\pi\right)\right]$

$= \sqrt{2}\left[-\frac{1}{\sqrt{2}} + i\left(-\frac{1}{\sqrt{2}}\right)\right] = -1 - i$

$z = 2\left[\cos\left(-\frac{1}{3}\pi\right) + i\sin\left(-\frac{1}{3}\pi\right)\right]$

$= 2\left[\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right] = 1 - \sqrt{3}i$

$wz = (-1-i)(1-\sqrt{3}i) = -1 + \sqrt{3}i - i + \sqrt{3}i^2 = (-1-\sqrt{3}) + (\sqrt{3}-1)i$

$Re(wz) = -1 - \sqrt{3}$ $\qquad$ $Imag(wz) = \sqrt{3} - 1$

$|wz| = 2\sqrt{2}$, $\arg(wz) = \frac{11}{12}\pi$

$\therefore wz = 2\sqrt{2}\left(\cos\frac{11}{12}\pi + i\sin\frac{11}{12}\pi\right)$

$\therefore 2\sqrt{2}\sin\frac{11}{12}\pi = \sqrt{3} - 1$

$\therefore \sin\frac{11}{12}\pi = \frac{\sqrt{3}-1}{2\sqrt{2}}$

$\sin\frac{1}{12}\pi = \sin\left(\pi - \frac{1}{12}\pi\right) = \sin\frac{11}{12}\pi = \frac{\sqrt{3}-1}{2\sqrt{2}}$

5(a) $u = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, $w = 4\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

$\Rightarrow |u| = 2$, $\arg u = \frac{\pi}{6}$, $|w| = 4$, $\arg(w) = -\frac{\pi}{3}$

$\therefore \left|\frac{u^*}{w^3}\right| = \frac{|u|}{|w|^3} = \frac{2}{4^3} = \frac{1}{32}$

$\therefore \arg\left(\frac{u^*}{w^3}\right) = -\arg u - 3\arg w$

$= -\frac{\pi}{6} - 3\left(-\frac{\pi}{3}\right) = \frac{5\pi}{6}$

(b) $z = -1 + i\sqrt{3}$

$z^2 + az = (-1+i\sqrt{3})^2 + a(-1+i\sqrt{3})$

$= 1 - 2\sqrt{3}i - 3 - a + a\sqrt{3}i$

$= (-2-a) + (2-a)\sqrt{3}i$

$\arg(z^2 + az) = -\frac{\pi}{2} \Rightarrow \tan^{-1}\frac{(2-a)\sqrt{3}}{-2-a} = -\frac{\pi}{2}$

$\Rightarrow \frac{(2-a)\sqrt{3}}{-2-a} = \tan\left(-\frac{\pi}{2}\right) \to -\infty$

$\Rightarrow -2 - a = 0$

$\therefore a = -2$

6. $q = \frac{e^{i\theta}}{1-e^{i\theta}} = \frac{e^{i\theta}}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}\right)}$

$= \frac{e^{i(\theta-\frac{\theta}{2})}}{-2i\sin\frac{\theta}{2}} = \frac{e^{i(\frac{\theta}{2})}}{-2i\sin\frac{\theta}{2}}$

$= \frac{\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}}{-2i\sin\frac{\theta}{2}}$

$= -\frac{1}{2i}\cot\frac{\theta}{2} \times \frac{i}{i} - \frac{1}{2}$

$= -\frac{1}{2} + \frac{1}{2}i\cot\frac{\theta}{2}$

(i) $\mathrm{Re}(q) = -\frac{1}{2}$

(ii) $\mathrm{Im}(q) = \frac{1}{2}\cot\frac{\theta}{2}$ (shown)

7. $|p| = \left|\frac{w}{w^*}\right| = \frac{|w|}{|w^*|} = \frac{r}{r} = 1$

$\arg\ p = \arg\left(\frac{w}{w^*}\right) = \arg w - \arg w^* = \theta - (-\theta) = 2\theta$

By De Moivre's Theorem, $p^5 = |p|^5 (\cos 2\theta + i \sin 2\theta)^5 = |p|^5 (\cos 10\theta + i \sin 10\theta)$

Since p^5 is real, $\sin 10\theta = 0$

Since p^5 is positive, $\cos 10\theta > 0$

Solving $\sin 10\theta = 0$, we obtain $\theta = 0, \frac{\pi}{10}, \frac{\pi}{5}, \frac{3\pi}{10}, \frac{2\pi}{5}$.

Since $\cos 10\theta > 0$, we have $\theta = \frac{\pi}{5}, \frac{2\pi}{5}$.

8. **(i)** $ww^* + 64\sqrt{3}\,i + 16iw = 0$

$(x+yi)(x-yi) + 64\sqrt{3}\,i + 16i(x+yi) = 0$

$x^2 + y^2 + 64\sqrt{3}\,i + 16ix - 16y = 0$

$(x^2 + y^2 - 16y) + (16x + 64\sqrt{3})i = 0$

Comparing coefficients,

$(16x + 64\sqrt{3}) = 0$ and $(x^2 + y^2 - 16y) = 0$

$x = -4\sqrt{3}$

$y^2 - 16y + 48 = 0$

$(y-4)(y-12) = 0$

$y = 4$ since $y < 5$

$\therefore w = -4\sqrt{3} + 4i$

(ii) $|w| = 8$, $\arg(w) = \frac{5\pi}{6}$, thus $w^n = 8^n\left(\cos\frac{5n\pi}{6} + i\sin\frac{5n\pi}{6}\right)$ **[M1]**

Since w^n is to real, $\mathrm{Im}(w^n) = 0$, $\frac{5n\pi}{6} = k\pi, k \in$ □ **[M1]**

$\Rightarrow n = \frac{6k}{5}, k \in$ □ **[A1]**

9	$z^2 + (a-\mathrm{i})z^* + 16 + b\mathrm{i} = 0$ $(2+3\mathrm{i})^2 + (a-\mathrm{i})(2-3\mathrm{i}) + 16 + b\mathrm{i} = 0$ $\Rightarrow -5 + 12\mathrm{i} + 2a - 3a\mathrm{i} - 2\mathrm{i} - 3 + 16 + b\mathrm{i} = 0$ $\Rightarrow 8 + 2a + (10 - 3a + b)\mathrm{i} = 0$

	By comparing real and imaginary coefficient, Real: $8+2a=0 \Rightarrow a=-4$ (ans) $\text{Im}: 10-3a+b=0 \Rightarrow b=-22$ (ans)

10(i) When $z = 1$,

$$LHS = (1)^3 + a(1)^2 - a(1) - 1 = 0 = RHS$$ [B1]

Hence $z = 1$ is a root of the equation.

(ii) Since 2i is a root,

$$\therefore (2i)^3 + a(2i)^2 - a(2i) - 1 = 0$$ [M1]

$$-8\text{i} - 4a - 2a\text{i} - 1 = 0$$

$$a(-4-2\text{i}) = 1+8\text{i}$$

$$a = \frac{1+8\text{i}}{-4-2\text{i}} = -1 - \frac{3}{2}\text{i}$$ [A1]

(iii) Let $z = \alpha$ be the other root.

$$(z-\alpha)(z-1)(z-2\text{i}) = z^3 + az^2 - az - 1$$ [M1]

Comparing the constant term, we have

$$-2\alpha\text{i} = -1 \quad \Rightarrow \alpha = -\frac{1}{2}\text{i}$$ [M1, A1]

11 **(i)** $z^2 + (6-2\text{i})z = (z+3-\text{i})^2 - (3-\text{i})^2$

$$= (z-(-3+\text{i}))^2 - (8-6\text{i})$$

$u = -3+\text{i}$ and $v = 8-6\text{i}$

(ii) By GC, $\sqrt{7-24\text{i}} = 4-3\text{i}$ or $-4+3\text{i}$

$$z^2 + (6-2\text{i})z = -1-18\text{i}$$

$$(z+(3-\text{i}))^2 - (8-6\text{i}) = -1-18\text{i}$$

$$(z+(3-\text{i}))^2 = 7-24\text{i}$$

$z+(3-\text{i}) = 4-3\text{i}$ or $z+(3-\text{i}) = -4+3\text{i}$

$z = 1-2\text{i}$ or $z = -7+4\text{i}$