Differential Equations

2 Solution

$V = Ax \quad \Rightarrow \quad \frac{dV}{dt} = A\frac{dx}{dt}$

$\frac{dV}{dt} = 5 - kx$, where k is a positive constant.

$\Rightarrow A\frac{dx}{dt} = 5 - kx$ (shown)

$A\frac{dx}{dt} = 5 - kx$

$\Rightarrow A\int \frac{1}{5-kx}\,dx = \int dt$

$\Rightarrow -\frac{A}{k}\ln|5-kx| = t + c_1$, where c_1 is an arbitrary constant

$\Rightarrow \ln|5-kx| = -\frac{k}{A}t + c$, where c is an arbitrary constant

$\Rightarrow |5-kx| = e^{-\frac{k}{A}t+c}$

$\Rightarrow 5 - kx = Be^{-\frac{k}{A}t}$, where B is an arbitrary constant

$\Rightarrow x = \frac{1}{k}(5 - Be^{-\frac{k}{A}t})$

When $t = 0,\ x = 0 \quad \Rightarrow B = 5$

$\therefore\ x = \frac{5}{k}(1 - e^{-\frac{k}{A}t})$

3 Solution

(i) $\alpha = 2\theta$

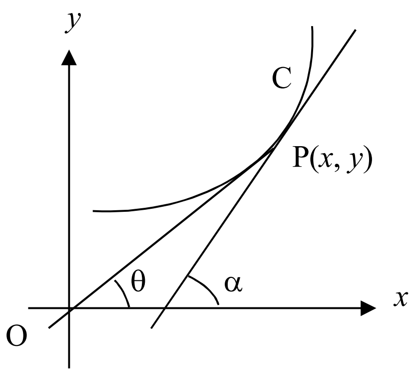
$\tan\alpha = \tan 2\theta$

$\frac{dy}{dx} = \tan 2\theta$

$= \frac{2\tan\theta}{1-\tan^2\theta}$

$= \frac{2\frac{y}{x}}{1-(\frac{y}{x})^2}$

$= \frac{2xy}{x^2-y^2}$



(ii) $\frac{dy}{dx} = \frac{2xy}{x^2-y^2}$ …… (1)

$y = vx$

$\frac{dy}{dx} = x\frac{dv}{dx} + v$ substitute into (1),

$x\frac{dv}{dx} + v = \frac{2x^2 v}{x^2 - v^2 x^2} = \frac{2v}{1-v^2}$

$x\frac{dv}{dx} = \frac{2v}{1-v^2} - v = \frac{2v - v + v^3}{1-v^2}$

$x\frac{dv}{dx} = \frac{v(1+v^2)}{1-v^2}$

(iii) $\frac{1-v^2}{v(1+v^2)} = \frac{a}{v} + \frac{b+cv}{1+v^2}$

$1 - v^2 = a(1+v^2) + v(b+cv)$

Comparing coefficients, $a = 1$, $b = 0$, $c = -2$

$\frac{1-v^2)}{v(1+v^2)} = \frac{1}{v} - \frac{2v}{1+v^2}$

$x\frac{dv}{dx} = \frac{v(1+v^2)}{1-v^2}$

$\int \frac{1-v^2}{v(1+v^2)} dv = \int \frac{1}{x} dx$

$\int \frac{1}{v} - \frac{2v}{1+v^2} dv = \int \frac{1}{x} dx$

$\ln|v| - \ln|1+v^2| = \ln|x| + \ln c$

$\ln\left|\frac{v}{1+v^2}\right| = \ln(c|x|)$

$\left|\frac{v}{1+v^2}\right| = c|x|$

$\frac{v}{1+v^2} = ax$ where $a = \pm c$

$v = ax(1+v^2)$

$\frac{y}{x} = ax(1+\frac{y^2}{x^2})$

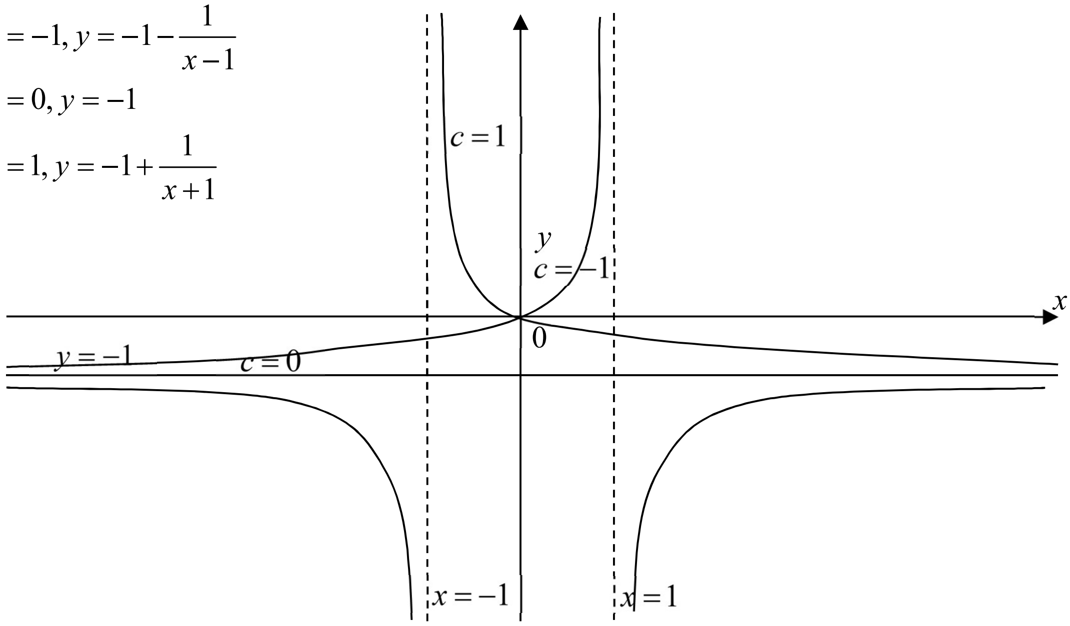
$y = a(x^2 + y^2)$

$x^2 + y^2 - \frac{y}{a} = 0$

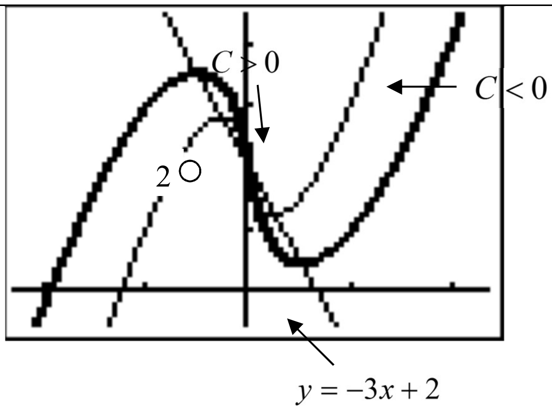
$x^2 + (y - \frac{1}{2a})^2 = \left(\frac{1}{2a}\right)^2$

The general solution represents a circle with center at $(0, \frac{1}{2a})$ and radius $\frac{1}{2a}$ units.

4 Solution

2	$x\dfrac{\mathrm{d}y}{\mathrm{d}x}=y^2+y$ use $y=ux$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}u}{\mathrm{d}x}x+u$ $\therefore x\left(\dfrac{\mathrm{d}u}{\mathrm{d}x}x+u\right)=(ux)^2+ux$ $\dfrac{\mathrm{d}u}{\mathrm{d}x}x+u=u^2x+u$ $\dfrac{1}{u^2}\dfrac{\mathrm{d}u}{\mathrm{d}x}=1$ $\int\dfrac{1}{u^2}\,\mathrm{d}u=\int 1\,\mathrm{d}x$ $-\dfrac{1}{u}=x+c$, where c is an arbitrary constant $-\dfrac{x}{y}=x+c$ $y=-\dfrac{x}{x+c}=-1+\dfrac{c}{x+c}$
2	Family of solution curves: $c=-1, y=-1-\dfrac{1}{x-1}$ $c=0, y=-1$ $c=1, y=-1+\dfrac{1}{x+1}$ 

5 Solution

	$y = vx \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = v + x\dfrac{\mathrm{dv}}{\mathrm{dx}}$ $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x + y - 2$ $\Rightarrow x(v + x\dfrac{\mathrm{dv}}{\mathrm{dx}}) = 3x + vx - 2$ $\Rightarrow \dfrac{\mathrm{dv}}{\mathrm{dx}} = \dfrac{3x-2}{x^2}$ $\Rightarrow \int \mathrm{d}v = \int \dfrac{3}{x} - \dfrac{2}{x^2}\,\mathrm{d}x$ $\Rightarrow v = 3\ln\lvert x\rvert + \dfrac{2}{x} + C$ $\Rightarrow y = 3x\ln\lvert x\rvert + 2 + Cx$
(i)	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow y = -3x + 2$
(ii)	(ii)  Note: (0,2) satisfies the differential equation. (0,2) is a singular solution.

6 Solution

Let the woman's jogging speed be v m/s.

$v \propto x \Rightarrow v = kx,$ for some constant k

$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = v - a = kx - a$

Given when $x = 1,\ \frac{\mathrm{d}x}{\mathrm{d}t} = 0 \Rightarrow k = a$

$\therefore \frac{\mathrm{d}x}{\mathrm{d}t} = ax - a = a(x-1)$ (shown)

$\Rightarrow \int \frac{1}{x-1}\mathrm{d}x = \int a\mathrm{d}t$

$\Rightarrow \ln|x-1| = at + c,\ c$ is a constant.

$\Rightarrow |x-1| = e^c.e^{at}$

$\Rightarrow x - 1 = Ae^{at}$, where $A = \pm e^c$

When $t = 0, x = 0 \Rightarrow A = -1$

$\therefore x - 1 = -e^{at} \Rightarrow x = 1 - e^{at}$

When $t = 2,\ x = 1 - e^{2a}$

$\Rightarrow e^{-2} = a\left[\left(1-e^{2a}\right)-1\right]$

$\Rightarrow e^{-2} = -ae^{2a}$

$\Rightarrow a = -1$ (by observation)

When $t = 4, x = 1 - e^{-4}$

Distanced required = $1 - e^{-4} - \left(1-e^{-2}\right) = e^{-2} - e^{-4}$

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7 Solution

$\frac{dV}{dt} = 300 - kV \Rightarrow \int \frac{1}{300-kV} dV = \int 1\, dt$

$\Rightarrow -\frac{1}{k}\ln|300-kV| = t + C$

$\Rightarrow |300-kV| = e^{-k(t+C)}$

$\Rightarrow 300 - kV = Ae^{-kt}$

When $t = 0$, $V = 0$ $\Rightarrow 300 - 0 = Ae^0 \Rightarrow A = 300$ $\Rightarrow V = \frac{300(1-e^{-kt})}{k}$ (Shown).

When $t = 20$, $V = 4500$, $\Rightarrow 4500 = \frac{300(1-e^{-20k})}{k}$ $\Rightarrow 15k = (1-e^{-20k})$

From the GC, $k = 0.030293$

2nd alarm : when $V = 6000$ $\therefore 6000 = \frac{300(1-e^{-0.030293t})}{0.030293}$ $\Rightarrow t = 30.7$

The residents will have **10.7 minutes** between the 1st and 2nd alarm.

$t \to \infty \Rightarrow V \to \frac{300}{0.030293} = 9903\ m^3$ which is impossible as the canal has only a fixed volume of $6000\ m^3$. The model is not valid for large values of t.

8 Solution

10(i)	Let the constant be a. $\therefore\ \frac{\mathrm{d}r}{\mathrm{d}t} = k(r-a)$, where k is a constant. Given $r = 43$ when $\frac{\mathrm{d}r}{\mathrm{d}t} = 0$, $\therefore\ 0 = k(43-a)$ Since $k \neq 0$, then $a = 43$ $\therefore\ \frac{\mathrm{d}r}{\mathrm{d}t} = k(r-43)$ (shown)
(ii)	$\int \frac{1}{r-43}\ \mathrm{d}r = k\int \mathrm{d}t$ $\ln\lvert r-43\rvert = kt + C_1$ $\lvert r-43\rvert = \mathrm{e}^{kt+C_1}$ $r = 43 + A\mathrm{e}^{kt}$ where $A = \mathrm{e}^{C_1}$ When $t = 0$, $r = 348$. $\therefore\ A = 305$. $\therefore\ r = 43 + 305\mathrm{e}^{kt}$
(iii)	$I = \int r\ \mathrm{d}t$ $= \int (43 + 305\mathrm{e}^{kt})\ \mathrm{d}t$ $= 43t + \frac{305}{k}\mathrm{e}^{kt} + C_2$ When $t = 0$, $I = 0$. $\therefore\ C_2 = -\frac{305}{k}$ $\therefore\ I = 43t + \frac{305}{k}(\mathrm{e}^{kt} - 1)$
(iv)	Given $I = 5700$ and $t = 90$, $\therefore\ 5700 = 43(90) + \frac{305}{k}(\mathrm{e}^{90k} - 1)$ $1830 = \frac{305}{k}(\mathrm{e}^{90k} - 1)$ $6k = \mathrm{e}^{90k} - 1$ Solving using GC, $k = -0.167$ or $k = 0$ (NA)
(v)	$r = 43 + 305\mathrm{e}^{-\frac{1}{6}t}$

	If t becomes larger, $305e^{-\frac{1}{6}t} \to 0$, $r \to 43$ Hence r would be reduced to a steady 43 kilobytes per second in the long run.

9 Solution

$\frac{d\theta}{dt} = -k(\theta - 30), \quad k > 0$

$\int \frac{1}{\theta - 30}\ d\theta = -\int k\ dt$

$\ln|\theta - 30| = -kt + C$

$|\theta - 30| = e^{-kt+C}$

$\theta - 30 = \pm e^{-kt}e^{C} = Ae^{-kt}$ (where $A = \pm e^{C}$)

$\theta = 30 + Ae^{-kt}$

when $t = 0$, $\theta = 90$

$90 = 30 + A$

$A = 60$

$\therefore \theta = 30 + 60e^{-kt}$ (shown)

when $t = 8$, $\theta = 55$

$55 = 30 + 60e^{-8k}$

$e^{-8k} = \frac{5}{12}$

$k = -\frac{1}{8}\ln\frac{5}{12}$

when $\theta = 35$,

$35 = 30 + 60e^{\left(\frac{1}{8}\ln\frac{5}{12}\right)t}$

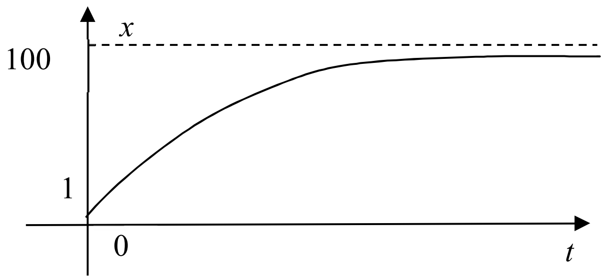
$e^{\left(\frac{1}{8}\ln\frac{5}{12}\right)t} = \frac{1}{12}$

$t = 22.707$

$\therefore$ additional time needed $= 22.707 - 8 = 14.7$ min (1d.p.)

10 Solution

(i)	$\frac{dx}{dt} = k(x)(100 - x)$ At $t = 0$, $x = 1, \frac{dx}{dt} = 1$ $1 = k(1)(99) \Rightarrow k = \frac{1}{99}$ therefore $\frac{dx}{dt} = \frac{1}{99}(100x - x^2)$

(ii)	$\frac{dx}{dt}=\frac{1}{99}(100x-x^2)$ $\int \frac{1}{x(100-x)}\,\mathrm{d}x=\int \frac{1}{99}\,\mathrm{d}t$ $\frac{1}{100}\int \frac{1}{x}+\frac{1}{100-x}\,\mathrm{d}x=\int \frac{1}{99}\,\mathrm{d}t$ $\ln x-\ln(100-x)=\frac{100}{99}t+C$ $\ln\left(\frac{x}{100-x}\right)=\frac{100}{99}t+C$ $\frac{x}{100-x}=e^{\frac{100}{99}t+C}=Ae^{\frac{100}{99}t}$ When $t=0$, $x=1$, $A=\frac{1}{99}$ $x=A100e^{\frac{100}{99}t}-xAe^{\frac{100}{99}t}$ $x=\frac{A100e^{\frac{100}{99}t}}{1+Ae^{\frac{100}{99}t}}=\frac{100e^{\frac{100}{99}t}}{99+e^{\frac{100}{99}t}}$
(iii)	
(iv)	Using GC, 5.64 years.
(v)	The farmers may be influenced by adoption of innovation from other sources, eg mass media, besides farmers. Or any other reasonable answer