Differential Equations

1 $\frac{dG}{dt} = \frac{G+1}{2}$

$\int \frac{1}{G+1} dG = \int \frac{1}{2} dt$

$\ln|G+1| = 0.5t + C$

$G+1 = \pm e^{0.5t+C}$

$G = -1 + Ae^{0.5t}$, where $A = \pm e^{C}$

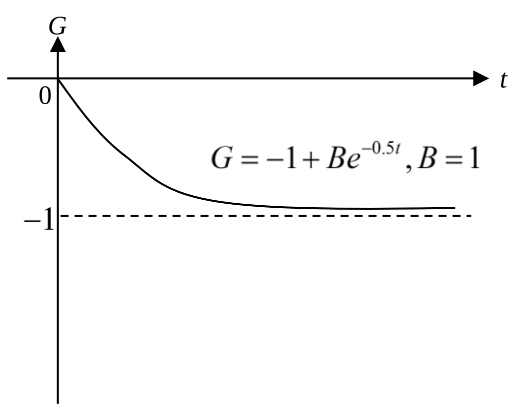
When $t = 0$, $G = 0$,

$A = 1$

$\therefore G = -1 + e^{0.5t}$

Examples of possible comments:
The model is not suitable because …

- ✓ The economist is assuming that that there are no fluctuations in the economic growth in the future.
- ✓ The economist is assuming that the country will enjoy perpetual economic growth in the long term.
- ✓ The economist is assuming Country *A* is always experiencing positive and increasing economic growth in the future.
- ✓ Factors affecting economic growth remains unchanged.



In the long term, Country *B* is expected to be still in recession with an economic growth decreasing towards -1%.

2 CJC/II/4

(a) Since $y = x$ and $\frac{dy}{dx} = 1$,

$LHS = 1 = \frac{x^2 + x^2}{2x^2} = RHS$

(b) $y = ux$

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$u + x\frac{du}{dx} = \frac{x^2 + u^2x^2}{2x(ux)}$

$u + x\frac{du}{dx} = \frac{1+u^2}{2u}$

$$x\frac{du}{dx}=\frac{1+u^2}{2u}-u$$

$$x\frac{du}{dx}=\frac{1+u^2-2u^2}{2u}$$

$$x\frac{du}{dx}=\frac{1-u^2}{2u}(shown)$$

$$\int\frac{2u}{1-u^2}\,du=\int\frac{1}{x}\,dx$$

$$\int\frac{-2u}{1-u^2}\,du=-\int\frac{1}{x}\,dx$$

$$\ln(1-u^2)=-\ln x+C$$

$$1-u^2=\frac{A}{x}$$

$$u^2=1-\frac{A}{x}$$

$$\frac{y^2}{x^2}=1-\frac{A}{x}$$

$$y^2=x^2-Ax$$

(c)

$$\frac{d^2x}{dt^2}=4ae^{-2t}$$

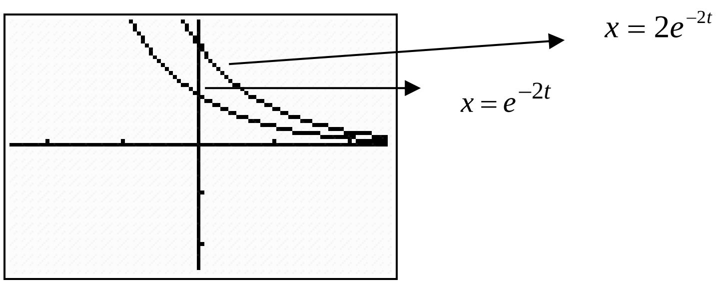
$$\frac{dx}{dt}=-2ae^{-2t}+C$$

$$x=ae^{-2t}+Ct+D$$

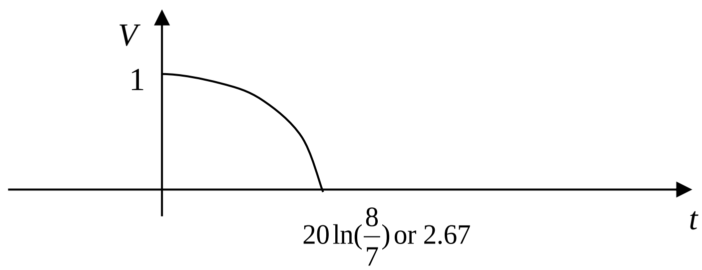
Since entire population is wiped out by the disease eventually, as $t\to+\infty, x\to 0$.
Hence, $C = 0$, $D = 0$.

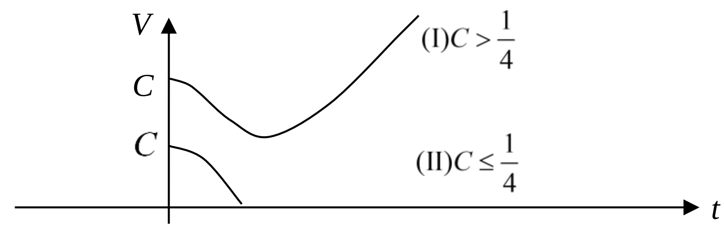
$\therefore x=ae^{-2t}$

a represents the initial population of the fish (in thousands).



3 DHS/II/3

3(i)	$\frac{dV}{dt}=\frac{1}{60}(V-\frac{8}{V^2})=\frac{1}{60}(\frac{V^3-8}{V^2})$ $\Rightarrow \frac{V^2}{V^3-8}\frac{dV}{dt}=\frac{1}{60}$ $\Rightarrow \frac{1}{3}\ln\lvert V^3-8\rvert=\frac{1}{60}t+C'$ $\Rightarrow \lvert V^3-8\rvert=e^{\frac{t}{20}+C''},\quad C''=3C'$ $\Rightarrow V^3-8=Ae^{\frac{t}{20}},\quad A=\pm e^{C''}$ When $t=0,\ V=1,$ $\Rightarrow A=-7,$ $\Rightarrow V=\sqrt[3]{8-7e^{\frac{t}{20}}}$ 

(ii)	$\dfrac{\mathrm{d}^2V}{\mathrm{d}t^2} = 12t^2 - 2$ $\Rightarrow \quad \dfrac{\mathrm{d}V}{\mathrm{d}t} = 4t^3 - 2t + C_1$ When $t = 0,\ \dfrac{\mathrm{d}V}{\mathrm{d}t} = 0,\ \therefore\ C_1 = 0$ $\Rightarrow \quad \dfrac{\mathrm{d}V}{\mathrm{d}t} = 4t^3 - 2t$ $\Rightarrow \quad V = t^4 - t^2 + C, C$ is a constant. $V = t^4 - t^2 + C$ $= (t^2 - \frac{1}{2})^2 + (C - \frac{1}{4})$ 
(iii)	When $t = 0,\ V = 1,$ then $C_2 = 1 > \frac{1}{4}$. Therefore given the above initial condition, Bob's model corresponds to solution curve type (I) in part (ii). Therefore in Bob's model, the volume of water approaches infinity in the long run (not realistic) whereas in Andy's model, the volume of water reasonably diminishes to zero in the long run/after some time. Thus, Andy's model is more appropriate than Bob's model.

4 MI/I/6

$$\frac{dx}{dt} = kx\left(N - x\right),\ k \text{ is a positive constant.}$$

$$\int \frac{1}{x(N-x)}\,dx = \int k\,dt$$

$$\frac{1}{N}\int \left[\frac{1}{x} + \frac{1}{N-x}\right] dx = kt + C$$

$$\ln x - \ln(N - x) = Nkt + NC$$

$$\ln\left(\frac{x}{N-x}\right) = Nkt + NC$$

$$\frac{x}{N-x} = Ae^{Nkt},\quad A = e^{NC}$$

When $t = 0$, $x = 1$, $\dfrac{1}{N-1} = A$

$$\frac{x}{N-x} = \frac{1}{N-1}e^{Nkt}$$

$$\Rightarrow x\left(N-1+e^{Nkt}\right) = Ne^{Nkt}$$

$\therefore x = \dfrac{Ne^{Nkt}}{N-1+e^{Nkt}}$ (shown)

5 RVHS/II/4

$$\frac{dl}{dt} = kl$$

$$\int \frac{1}{l}\,dl = \int k\,dt$$

$\ln l = kt + C$ (note: $l > 0$)

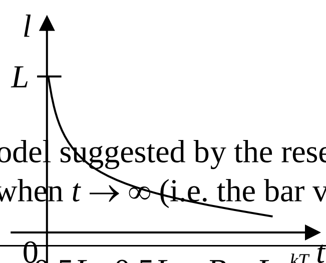
$l = e^{kt+C}$

$l = Ae^{kt}$ (shown)

$A > 0$ and $k < 0$

When $t = 0$, $l = L$: $A = L$

$\therefore\ l = Le^{kt}$



The model suggested by the researcher is not suitable as $l \to 0$ when $t \to \infty$ (i.e. the bar vanished).

$t = T, l = 0.5L : 0.5L = B + Le^{kT}$

$$\Rightarrow \frac{0.5L-B}{L} = e^{kT}$$

When $t = 3T$, $l = B + Le^{3kT} = B + \dfrac{(0.5L-B)^3}{L^2}$

6 SAJC/II/2

$$\frac{dy}{dt} = -y(y-1)$$

$$\int \frac{1}{y(y-1)}\, dy = \int -1\, dt$$

$$\int -\frac{1}{y} + \frac{1}{y-1}\, dy = \int -1\, dt$$

$$-\ln|y| + \ln|y-1| = -t + C$$

$$\ln\left|\frac{y-1}{y}\right| = -t + C$$

$$\left|\frac{y-1}{y}\right| = e^{-t+C} = Ae^{-t} \quad , A = e^{C}$$

$$\frac{y-1}{y} = Be^{-t} \qquad , B = \pm A$$

$$1 - \frac{1}{y} = Be^{-t}$$

$$\frac{1}{y} = 1 - Be^{-t}$$

$$y = \frac{1}{1 - Be^{-t}}$$

When $t = 2, \frac{950}{1000} = \frac{1}{1 - Be^{-2}}$

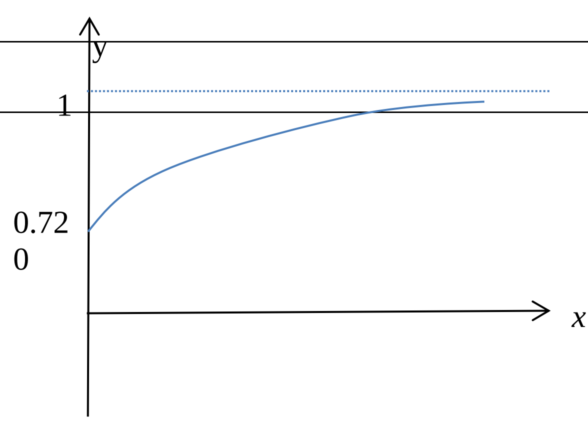
$$1 - Be^{-2} = \frac{20}{19} \Rightarrow B = -\frac{1}{19}e^{2} = -0.38890$$

Therefore $\quad y = \dfrac{1}{1 - \left(-0.3889e^{0}\right)} = 0.71999$

Thus the original population is 720.

As $t \to \infty,\ Be^{-t} \to 0$.

Then $y \to 1$, i.e. the population size **<u>increases to 1000</u>** in the long run.



7 SRJC/II/5

$$\frac{dx}{dt} \propto x(35-x)$$

$$\Rightarrow \frac{dx}{dt} = kx(35-x), k>0$$

$$\Rightarrow \int \frac{1}{x(35-x)} dx = \int k \, dt$$

$$\Rightarrow \frac{1}{35}\int \left(\frac{1}{x} + \frac{1}{35-x}\right) dx = k\int dt$$

$$\Rightarrow \ln x - \ln(35-x) = 35kt + c$$

$$\Rightarrow \ln\left(\frac{x}{35-x}\right) = 35kt + C$$

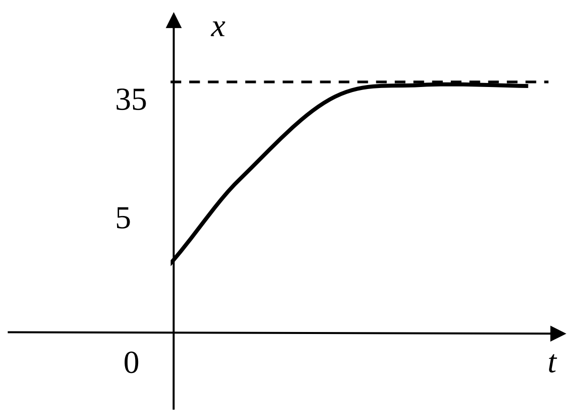
$$\Rightarrow \frac{x}{35-x} = e^{35kt+c} = Ae^{35kt}, \; A>0$$

$$\Rightarrow x = \frac{35Ae^{35kt}}{1+Ae^{35kt}}$$

When t=0, x=5, $\Rightarrow A = \frac{1}{6}$

$$x = 35\left[1 - \frac{6}{6+e^{35kt}}\right]$$

As $t \to \infty, x \to 35$



[1 mark – for crucial points and 1 mark – shape]

$$\frac{dx}{dt} \propto (A - x)$$

$$\Rightarrow \frac{dx}{dt} = k(A - x)\ , \ k > 0$$

$$\int \frac{1}{A - x}\ dx = \int k\ \ dt$$

$-\ln(A - x) = kt + \mathrm{C}$ where C is a constant

$\ln(A - x) = -kt - \mathrm{C}$

$A - x = e^{-kt}(B)$ where $B = e^{-\mathrm{C}}$ is a constant

$\therefore\ x = A - B\, e^{-kt}$

When $t = 0$, $x = 0 \ \Rightarrow B = A$

When $t = 2 \ln 2$, $x = \frac{1}{2} A$

$$\Rightarrow \frac{1}{2} A = A - Ae^{-k(2\ln 2)}$$

$$\Rightarrow e^{-k(2\ln 2)} = \frac{1}{2}$$

$$\Rightarrow -k(2\ln 2) = \ln(\frac{1}{2}) = -\ln 2$$

$$\Rightarrow k = \frac{1}{2}$$

$$\therefore x = A(1 - e^{-\frac{1}{2}t}) \quad //$$

When $x = \dfrac{3}{4} A$,

$$\frac{3}{4} A = A(1 - e^{-\frac{1}{2}t})$$

$$\Rightarrow e^{-\frac{1}{2}t} = \frac{1}{4}$$

$$\Rightarrow -\frac{1}{2} t = \ln(\frac{1}{4}) = -2\ln 2$$

$\Rightarrow t = 4 \ln 2$

Additional time required = 4 ln2 – 2 ln2

= (2 ln 2) minutes //