Discrete Random Variables

1 (i) $E(X)=4 \quad \Rightarrow \quad \sum xP(X=x)=2p+3\left(\frac{2}{10}\right)+4\left(\frac{3}{10}\right)+5q=4$

$$20p+50q=22 \qquad \text{----- (1)}$$

$\sum P(X=x)=1 \quad \Rightarrow \quad p+\frac{2}{10}+\frac{3}{10}+q=1$

$$2p+2q=1 \qquad \text{----- (2)}$$

Solving (1) and (2), $p=\frac{1}{10},\ q=\frac{2}{5}$

(ii) $E(X^2)=\sum x^2P(X=x)=2^2\left(\frac{1}{10}\right)+3^2\left(\frac{2}{10}\right)+4^2\left(\frac{3}{10}\right)+5^2\left(\frac{2}{5}\right)=17$

$\text{Var}(X)=E(X^2)-[E(X)]^2=17-4^2=1$ (shown)

(iii) $E(|X-4|)=\sum |x-4|P(X=x)$

$$=|2-4|\left(\frac{1}{10}\right)+|3-4|\left(\frac{2}{10}\right)+|4-4|\left(\frac{3}{10}\right)+|5-4|\left(\frac{2}{5}\right)$$

$$=\frac{2}{10}+\frac{2}{10}+0+\frac{2}{5}=\frac{4}{5}$$

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2 Let R denote the score on the red die and G be the score on the green die.

Product		R		
		1	2	3
G	1	1	2	3
	2	2	4	6

$P(Y=1)=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)=\frac{1}{6}$ $\qquad P(Y=2)=2\left(\frac{1}{3}\times\frac{1}{2}\right)=\frac{1}{3}$

$P(Y=3)=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)=\frac{1}{6}$ $\qquad P(Y=4)=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)=\frac{1}{6}$

$P(Y=6)=\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)=\frac{1}{6}$

X	1	2	3	4	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E(X)=\sum xP(X=x)=1\left(\frac{1}{6}\right)+2\left(\frac{1}{3}\right)+3\left(\frac{1}{6}\right)+4\left(\frac{1}{6}\right)+6\left(\frac{1}{6}\right)=\frac{9}{3}$

$E(X^2)=\sum x^2P(X=x)=1^2\left(\frac{1}{6}\right)+2^2\left(\frac{1}{3}\right)+3^2\left(\frac{1}{6}\right)+4^2\left(\frac{1}{6}\right)+6^2\left(\frac{1}{6}\right)=\frac{35}{3}$

$\text{Var}(X)=E(X^2)-[E(X)]^2=\frac{35}{3}-\left(\frac{9}{3}\right)^2=\frac{8}{3}$

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3 $\mathrm{E}(X) = \sum x\mathrm{P}(X = x) = 0(0.1) + 2(0.2) + 4(0.3) + 6(0.4) = 4$

$\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X = x) = 0^2(0.1) + 2^2(0.2) + 4^2(0.3) + 6^2(0.4) = 20$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 20 - 4^2 = 4$ (shown)

$\mathrm{E}(X^4) = \sum x^4\mathrm{P}(X = x) = 0^4(0.1) + 2^4(0.2) + 4^4(0.3) + 6^4(0.4) = 598.4$

$\mathrm{Var}(X^2) = \mathrm{E}(X^4) - [\mathrm{E}(X^2)]^2 = 598.4 - 20^2 = 198.4$

$\mathrm{P}(X_1 + X_2 = 6)$

$= \mathrm{P}(X_1 = 0 \text{ and } X_2 = 6) + \mathrm{P}(X_1 = 2 \text{ and } X_2 = 4) + \mathrm{P}(X_1 = 4 \text{ and } X_2 = 2) + \mathrm{P}(X_1 = 6 \text{ and } X_2 = 0)$

$= \mathrm{P}(X_1 = 0)\mathrm{P}(X_2 = 6) + \mathrm{P}(X_1 = 2)\mathrm{P}(X_2 = 4) + \mathrm{P}(X_1 = 4)\mathrm{P}(X_2 = 2) + \mathrm{P}(X_1 = 6)\mathrm{P}(X_2 = 0)$

$= 2(0.1)(0.4) + 2(0.2)(0.3)$

$= 0.2$ (shown)

Let $Y = X_1 + X_2$

+		X_1			
		0	2	4	6
X_2	0	0	2	4	6
	2	2	4	6	8
	4	4	6	8	10
	6	6	8	10	12

$\mathrm{P}(Y = 0) = (0.1)^2 = 0.01$

$\mathrm{P}(Y = 2) = 2(0.1)(0.2) = 0.04$

$\mathrm{P}(Y = 4) = 2(0.1)(0.3) + (0.2)^2 = 0.10$

$\mathrm{P}(Y = 6) = 2(0.1)(0.4) + 2(0.2)(0.3) = 0.20$

$\mathrm{P}(Y = 8) = 2(0.2)(0.4) + (0.3)^2 = 0.25$

$\mathrm{P}(Y = 10) = 2(0.3)(0.4) = 0.24$

$\mathrm{P}(Y = 12) = (0.4)^2 = 0.16$

y	0	2	4	6	8	10	12
$\mathrm{P}(Y = y)$	0.01	0.04	0.10	0.20	0.25	0.24	0.16

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4 $\mathrm{E}(X) = \sum x\mathrm{P}(X = x) = (-2)\left(\frac{1}{4}\right) + 0\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = 0$ (or by symmetry)

$\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X = x) = (-2)^2\left(\frac{1}{4}\right) + 0^2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{4}\right) = 2$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = 2 - 0^2 = 2$

$\mathrm{E}(|X|) = \sum |x|\mathrm{P}(X = x) = (2)\left(\frac{1}{4}\right) + 0\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = 1$

$Y = X_1 + X_2$

+		X_1		
		-2	0	2
X_2	-2	-4	-2	0
	0	-2	0	2
	2	0	2	4

$\mathrm{P}(Y=-4) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$ $\qquad$ $\mathrm{P}(Y=-2) = 2\left(\frac{1}{4}\times\frac{1}{2}\right) = \frac{1}{4}$

$\mathrm{P}(Y=0) = 2\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{8}$ $\qquad$ $\mathrm{P}(Y=2) = 2\left(\frac{1}{4}\times\frac{1}{2}\right) = \frac{1}{4}$

$\mathrm{P}(Y=4) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

y	-4	-2	0	2	4
$\mathrm{P}(Y=y)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$\mathrm{Var}(Y) = \mathrm{Var}(X_1+X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 2+2 = 4$

$\mathrm{E}(Y+3) = \mathrm{E}(Y)+3 = \mathrm{E}(X_1+X_2)+3 = \mathrm{E}(X_1)+\mathrm{E}(X_2)+3 = 3$

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5 (a) $\sum \mathrm{P}(X=x) = 1 \quad \Rightarrow \quad p + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 1$

$$p = \frac{1}{16}$$

(b) $\mathrm{E}(X) = \sum x\mathrm{P}(X=x) = 0\left(\frac{1}{16}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + 8\left(\frac{1}{16}\right) = 2$

$\mathrm{E}(X^2) = \sum x^2\mathrm{P}(X=x) = 0^2\left(\frac{1}{16}\right) + 1^2\left(\frac{1}{2}\right) + 2^2\left(\frac{1}{4}\right) + 4^2\left(\frac{1}{8}\right) + 8^2\left(\frac{1}{16}\right) = \frac{15}{2}$

$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2 = \frac{15}{2} - 2^2 = \frac{7}{2}$

(c) $\mathrm{E}(2X+3) = 2\mathrm{E}(X)+3 = 2(2)+3 = 7$

$\mathrm{Var}(2X+3) = 4\mathrm{Var}(X) = 4\left(\frac{7}{2}\right) = 14$

(d) $\mathrm{E}(X_1+X_2) = \mathrm{E}(X_1)+\mathrm{E}(X_2) = 2\mathrm{E}(X) = 4$

$\mathrm{Var}(X_1 - 7X_2) = \mathrm{Var}(X) + 49\mathrm{Var}(X) = 50\mathrm{Var}(X) = 50(\frac{7}{2}) = 175$

$\mathrm{P}(X_1+X_2=2)$

$= \mathrm{P}(X_1=0 \text{ and } X_2=2) + \mathrm{P}(X_1=1 \text{ and } X_2=1) + \mathrm{P}(X_1=2 \text{ and } X_2=0)$

$= \mathrm{P}(X_1=0)\mathrm{P}(X_2=2) + \mathrm{P}(X_1=1)\mathrm{P}(X_2=1) + \mathrm{P}(X_1=2)\mathrm{P}(X_2=0)$

$= 2\left(\frac{1}{16}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)^2 = \frac{9}{32}$

6

		Die					
		1	2	3	4	5	6
Coin	H	1	2	3	4	5	6
	T	2	4	6	8	10	12

x	1	2	3	4	5	6	8	10	12
$P(X=x)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$

$$\mu = E(X) = \sum xP(X=x)$$
$$= \frac{1}{12}(1+3+5+8+10+12)+\frac{2}{12}(2+4+6) = \frac{21}{4}$$

$$P(X<\mu) = P\left(X < 5\frac{1}{4}\right) = P(X = 1, 2, 3, 4 \text{ or } 5)$$
$$= 3\left(\frac{1}{12}\right)+2\left(\frac{2}{12}\right) = \frac{7}{12} \qquad \text{(shown)}$$

$$E(X^2) = \sum x^2 P(X=x)$$
$$= \frac{1}{12}(1^2+3^2+5^2+8^2+10^2+12^2)+\frac{2}{12}(2^2+4^2+6^2)$$
$$= \frac{455}{12}$$

$$\text{Var}(X) = E(X^2)-[E(X)]^2$$
$$= \frac{455}{12}-\left(\frac{21}{4}\right)^2$$
$$= \frac{497}{48}$$

Let $Y = X_1 + X_2$

$$P(Y=4) = P(X_1=1)P(X_2=3)+P(X_1=2)P(X_2=2)+P(X_1=3)P(X_2=1) = \frac{1}{24}$$

$$E(Y) = E(X_1)+E(X_2) = 2E(X) = 2\left(\frac{21}{4}\right) = \frac{21}{2}$$

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7 Let A and B be the scores on Card 1 and Card 2 respectively.

(a) $P(A=1 \text{ and } B=1) = P(A=1)P(B=1) = \frac{135}{360}\times\frac{180}{360} = \frac{3}{16}$

(b) $P(\text{at least one card is } 3) = 1-P(\text{no card is } 3) = 1-\left(\frac{225}{360}\times\frac{270}{360}\right) = \frac{17}{32}$

$$P(X=2) = P(A=2 \text{ and } B=1)+P(A=2 \text{ and } B=2)+P(A=1 \text{ and } B=2)$$
$$= \frac{90}{360}\left(\frac{180}{360}\right)+\frac{90}{360}\left(\frac{90}{360}\right)+\frac{135}{360}\left(\frac{90}{360}\right)$$
$$= \frac{9}{32} \qquad \text{(shown)}$$

$$P(X=1) = P(A=1 \text{ and } B=1) = \frac{135}{360}\left(\frac{180}{360}\right) = \frac{3}{16}$$

$$P(X=3)=1-[P(X=1)+P(X=2)]=1-\frac{9}{32}-\frac{3}{16}=\frac{17}{32}$$

$$E(X)=\sum xP(X=x)=1\left(\frac{3}{16}\right)+2\left(\frac{9}{32}\right)+3\left(\frac{17}{32}\right)=\frac{75}{32}$$ (shown)

$$E(X^2)=\sum x^2P(X=x)=1^2\left(\frac{3}{16}\right)+2^2\left(\frac{9}{32}\right)+3^2\left(\frac{17}{32}\right)=\frac{195}{32}$$

$$\text{Var}(X)=E(X^2)-[E(X)]^2=\frac{195}{32}-\left(\frac{75}{32}\right)^2=\frac{615}{1024}$$

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8 Let Y denote the score on a die.
Let $S=Y_1+Y_2$.

$$P(S\le 4)$$
$$=P(S=2)+P(S=3)+P(S=4)$$
$$=P(Y_1=1,Y_2=1)+[P(Y_1=1,Y_2=2)+P(Y_1=2,Y_2=1)]$$
$$+[P(Y_1=1,Y_2=3)+P(Y_1=2,Y_2=2)+P(Y_1=3,Y_2=1)]$$
$$=\left(\frac{1}{6}\right)^2+2\left(\frac{1}{6}\right)^2+3\left(\frac{1}{6}\right)^2$$
$$=\frac{1}{6}$$
$$\therefore P(S\ge 5)=1-P(S\le 4)=1-\frac{1}{6}=\frac{5}{6}$$

Let A be Alfred's **gain** after one game.

a	$x+8$	$-x$
$P(A=a)$	$\frac{1}{6}$	$\frac{5}{6}$

$$E(A)=\sum aP(A=a)=(x+8)\left(\frac{1}{6}\right)+(-x)\left(\frac{5}{6}\right)=\frac{4}{3}-\frac{2}{3}x$$

Alfred's expected cash after one game $=100+\left(\frac{4}{3}-\frac{2}{3}x\right)=\frac{1}{3}(300+4-2x)=\$\frac{1}{3}(304-2x)$

(shown)

$$E(A_1+A_2+A_3+A_4+A_5+A_6)=6E(A)=6\left(\frac{4}{3}-\frac{2}{3}x\right)=8-4x$$

Alfred's expected cash after six games $=100+(8-4x)=\$(108-4x)$

Let B be Bertie's gain after one game.

b	$-(x+8)$	x

$P(B=b)$	$\frac{1}{6}$	$\frac{5}{6}$

$$E(B)=\sum bP(B=b)=-(x+8)\left(\frac{1}{6}\right)+x\left(\frac{5}{6}\right)=-\frac{4}{3}+\frac{2}{3}x$$

For game to be fair, $E(A)=E(B)$

$$\frac{4}{3}-\frac{2}{3}x=-\frac{4}{3}+\frac{2}{3}x \quad \Rightarrow x=2$$

Alternative (shorter) method:

For game to be fair,

$$E(A)=0$$

$$\frac{4}{3}-\frac{2}{3}x=0$$

$$x=2$$

Given that $x=3$,

a	11	-3
$P(A=a)$	$\frac{1}{6}$	$\frac{5}{6}$

$$E(A)=\frac{4}{3}-\frac{2}{3}(3)=-\frac{2}{3}$$

$$E(A^2)=\sum a^2P(A=a)=11^2\left(\frac{1}{6}\right)+(-3)^2\left(\frac{5}{6}\right)=\frac{83}{3}$$

$$Var(A)=E(A^2)-[E(A)]^2=\frac{83}{3}-\left(-\frac{2}{3}\right)^2=\frac{245}{9}$$

$$Var(100+A)=Var(A)=\frac{245}{9}=27\frac{2}{9}$$

9

r	0	1	2	3
$P(R=r)$	$\frac{n(n-1)(n-2)}{(n+3)(n+2)(n+1)}$	$\frac{3x3n(n-1)}{(n+3)(n+2)(n+1)}$ $=\frac{9n(n-1)}{(n+3)(n+2)(n+1)}$	$\frac{3x3x2xn}{(n+3)(n+2)(n+1)}$ $=\frac{18n}{(n+3)(n+2)(n+1)}$	$\frac{3.2.1}{(n+3)(n+2)(n+1)}$ $=\frac{6}{(n+3)(n+2)(n+1)}$

$$E(R)=\sum rP(R=r)=1\left(\frac{9n(n-1)}{(n+3)(n+2)(n+1)}\right)+2\left(\frac{18n}{(n+3)(n+2)(n+1)}\right)+3\left(\frac{6}{(n+3)(n+2)(n+1)}\right)=..=\frac{9}{n+3}$$

$$E(R^2)=\sum r^2P(R=r)=1\left(\frac{9n(n-1)}{(n+3)(n+2)(n+1)}\right)+2^2\left(\frac{18n}{(n+3)(n+2)(n+1)}\right)+3^2\left(\frac{6}{(n+3)(n+2)(n+1)}\right)$$

$$=..=\frac{9n^2+63n+54}{(n+3)(n+2)(n+1)}$$

$$\mathrm{Var}(R) = \mathrm{E}(R^2\ ) - \left[\mathrm{E}(R)\right]^2 = \frac{9n^2 + 63n + 54}{(n+3)(n+2)(n+1)} - \left(\frac{9}{n+3}\right)^2 = \ldots$$

$$= \frac{9n^2}{(n+3)^2(n+2)}$$

$$\mathrm{E}(W) = \mathrm{E}(3-R) = 3 - \mathrm{E}(R) = 3 - \frac{9}{n+3} = \frac{3n}{(n+3)}$$

$$\mathrm{Var}(W) = \mathrm{Var}(3-R) = \mathrm{Var}(R) = \frac{9n^2}{(n+3)^2(n+2)}$$
