

Integration Techniques - Solutions

$$1 \text{ (a)} \quad \int \frac{\cos 3x - \operatorname{cosec}^2 3x}{\sin 3x + \cot 3x} dx = \frac{1}{3} \ln |\sin 3x + \cot 3x| + C$$

$$\begin{aligned} \text{(b)} \quad \int \frac{1-x}{\sqrt{1-16x^2}} dx &= \int \frac{1}{\sqrt{1-(4x)^2}} dx - \left(\frac{-1}{32} \right) \int \frac{-32x}{(1-16x^2)^{\frac{1}{2}}} dx \\ &= \frac{\sin^{-1}(4x)}{4} + \frac{1}{16} \sqrt{1-16x^2} + C \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int (1-x)^{-2} \ln x \, dx &= (\ln x) \left(\frac{1}{1-x} \right) - \int \frac{1}{x(1-x)} dx \\ &= \frac{\ln x}{1-x} - \int \left(\frac{1}{1-x} + \frac{1}{x} \right) dx \\ &= \frac{\ln x}{1-x} + \ln |1-x| - \ln |x| + C \end{aligned}$$

$$\begin{aligned} 2 \text{ (a)} \quad \int \frac{x}{3+x} dx &= \int \frac{3+x-3}{3+x} dx \\ &= \int 1 - \frac{3}{3+x} dx \\ &= x - 3 \ln |3+x| + c \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad \int x^2 \ln x \, dx &= \left[\frac{x^3}{3} \ln x \right] - \int \frac{x^3}{3} \frac{1}{x} dx \\ &= \left[\frac{x^3}{3} \ln x \right] - \frac{1}{3} \int x^2 dx \\ &= \left[\frac{x^3}{3} \ln x \right] - \frac{x^3}{9} + C \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \int \frac{x+3}{x^2+4x+7} dx &= \frac{1}{2} \int \frac{2x+6}{x^2+4x+7} dx \\ &= \frac{1}{2} \int \frac{2x+4+2}{x^2+4x+7} dx \\ &= \frac{1}{2} \left[\int \frac{2x+4}{x^2+4x+7} dx + \int \frac{2}{x^2+4x+7} dx \right] \\ &= \frac{1}{2} \left[\ln |x^2+4x+7| + \int \frac{2}{(x+2)^2+3} dx \right] \\ &= \frac{1}{2} \left[\ln |x^2+4x+7| + \frac{2}{\sqrt{3}} \tan^{-1} \frac{x+2}{\sqrt{3}} \right] + C \end{aligned}$$

$$\begin{aligned} \text{(d) Let } x &= 2 \sin \theta \\ dx &= 2 \cos \theta \, d\theta \end{aligned}$$

$$\int \frac{2x-1}{\sqrt{4-x^2}} dx = \int \frac{4 \sin \theta - 1}{2 \cos \theta} 2 \cos \theta \, d\theta$$

$$= \int 4 \sin \theta - 1 \, d\theta$$

$$= -4 \cos \theta - \theta + c$$

$$= -4 \frac{\sqrt{4-x^2}}{2} - \sin^{-1} \frac{x}{2} + c$$

$$= -2 \sqrt{4-x^2} - \sin^{-1} \frac{x}{2} + c$$

3 (a) $A(2x+6)+b=2Ax+6A+b$

Comparing coefficients of x , $2A=1 \Rightarrow A=\frac{1}{2}$

Comparing coefficients of constant term, $3+B=4 \Rightarrow B=1$

$$\therefore x+4 = \frac{1}{2}(2x+6)+1$$

$$\int \frac{x+4}{x^2+6x+13} dx$$

$$= \int \frac{\frac{1}{2}(2x+6)+1}{x^2+6x+13} dx$$

$$= \frac{1}{2} \int \frac{2x+6}{x^2+6x+13} dx + \int \frac{1}{x^2+6x+13} dx$$

$$= \frac{1}{2} \int \frac{2x+6}{x^2+6x+13} dx + \int \frac{1}{(x+3)^2+2^2} dx$$

$$= \frac{1}{2} \ln|x^2+6x+13| + \frac{1}{2} \tan^{-1}\left(\frac{x+3}{2}\right) + c$$

(b) $x = \frac{1}{u}$

$$dx = -\frac{1}{u^2} du$$

When $x = 2$, $u = \frac{1}{2}$.

When $x = 4$, $u = \frac{1}{4}$.

$$\int_2^4 \frac{1}{x^3} e^x dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{4}} u^3 e^u \left(-\frac{1}{u^2} \right) du$$

$$= \int_{\frac{1}{2}}^{\frac{1}{4}} -ue^u du$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} ue^u du \text{ (shown)}$$

$$\int_2^4 \frac{1}{x^3} e^x dx$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} ue^u du$$

$$= \left[ue^u \right]_{\frac{1}{4}}^{\frac{1}{2}} - \int_{\frac{1}{4}}^{\frac{1}{2}} e^u du$$

$$= \left(\frac{1}{2} e^{\frac{1}{2}} - \frac{1}{4} e^{\frac{1}{4}} \right) - \left[e^u \right]_{\frac{1}{4}}^{\frac{1}{2}}$$

$$\begin{aligned}
&= \left(\frac{1}{2} e^{\frac{1}{2}} - \frac{1}{4} e^{\frac{1}{4}} \right) - \left(e^{\frac{1}{2}} - e^{\frac{1}{4}} \right) \\
&= \frac{3}{4} e^{\frac{1}{4}} - \frac{1}{2} e^{\frac{1}{2}}
\end{aligned}$$

(c) $|x+2| = \begin{cases} x+2 & \text{if } x \geq -2 \\ -(x+2) & \text{if } x < -2 \end{cases}$

$$\begin{aligned}
&\int_{-3}^0 |x+2|^3 \, dx \\
&= \int_{-3}^{-2} [-(x+2)]^3 \, dx + \int_{-2}^0 (x+2)^3 \, dx \\
&= - \left[\frac{(x+2)^4}{4} \right]_{-3}^{-2} + \left[\frac{(x+2)^4}{4} \right]_{-2}^0 \\
&= -\frac{1}{4}(0-1) + \frac{1}{4}(2^4-0) \\
&= \frac{17}{4}
\end{aligned}$$

4(a) **Method 1**

$$x = \sin t \Rightarrow \frac{dx}{dt} = \cos t$$

$$\begin{aligned}
\int \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx &= \int \frac{(\sin t) e^{\sin^{-1}(\sin t)}}{\sqrt{1-\sin^2 t}} \cos t \, dt \\
&= \int \frac{(\sin t) e^t}{\cos t} \cos t \, dt \\
&= \int e^t \sin t \, dt \text{ (shown)}
\end{aligned}$$

Method 2

$$x = \sin t \Rightarrow t = \sin^{-1} x \Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\begin{aligned} \int \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx &= \int x e^{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} dx \\ &= \int (\sin t) e^{\sin^{-1}(\sin t)} dt \\ &= \int e^t \sin t dt \text{ (shown)} \end{aligned}$$

4(a) **Method 1**

$$u = e^t, \quad v = \sin t$$

$$\frac{du}{dt} = e^t, \quad \int v dt = -\cos t$$

$$u = e^t, \quad v = \cos t$$

$$\frac{du}{dt} = e^t, \quad \int v dt = \sin t$$

$$\begin{aligned} \int e^t \sin t dt &= e^t(-\cos t) - \int e^t(-\cos t) dt \\ &= -e^t \cos t + \int e^t \cos t dt \\ &= -e^t \cos t + e^t \sin t - \int e^t \sin t dt \end{aligned}$$

$$\text{Hence, } \int e^t \sin t dt = \frac{1}{2} e^t (\sin t - \cos t) + c$$

$$\begin{aligned} \int \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx &= \int e^t \sin t dt \\ &= \frac{1}{2} e^t (\sin t - \cos t) + c \\ &= \frac{1}{2} e^{\sin^{-1} x} [x - \cos(\sin^{-1}(x))] + c \text{ OR} \\ &= \frac{1}{2} e^{\sin^{-1} x} (x - \sqrt{1-x^2}) + c \end{aligned}$$

Method 2

$$u = \sin t \quad v = e^t$$

$$\frac{du}{dt} = \cos t \quad \int v \, dt = e^t$$

$$u = \cos t \quad v = e^t$$

$$\frac{du}{dt} = -\sin t \quad \int v \, dt = e^t$$

$$\int e^t \sin t \, dt = e^t \sin t - \int e^t \cos t \, dt$$

$$= e^t \sin t - \left[e^t \cos t - \int e^t (-\sin t) \, dt \right]$$

$$= e^t \sin t - e^t \cos t - \int e^t \sin t \, dt$$

$$\text{Hence, } \int e^t \sin t \, dt = \frac{1}{2} e^t (\sin t - \cos t) + c$$

$$\int \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}} \, dx = \int e^t \sin t \, dt$$

$$= \frac{1}{2} e^t (\sin t - \cos t) + c$$

$$= \frac{1}{2} e^{\sin^{-1} x} (x - \sqrt{1-x^2}) + c$$

5.

$$t^3 = 3x + 4$$

$$3t^2 \frac{dt}{dx} = 3$$

$$\frac{dt}{dx} = \frac{1}{t^2}$$

$$t = \sqrt[3]{3x+4}$$

$$\frac{dt}{dx} = \frac{1}{3} (3x+4)^{-\frac{2}{3}} (3)$$

$$\frac{dt}{dx} = \frac{1}{(3x+4)^{\frac{2}{3}}}$$

$$\frac{dt}{dx} = \frac{1}{t^2}$$

$$\begin{aligned}
\int e^{\sqrt[3]{3x+4}} dx &= \int e^t t^2 dt \text{ (shown)} \\
&= e^t t^2 - \int e^t 2t dt \\
&= e^t t^2 - 2 \left[te^t - \int e^t dt \right] + C_1 \\
&= e^t t^2 - 2 \left[te^t - e^t \right] + C_2 \\
&= e^t t^2 - 2te^t + 2e^t + C_2 \\
&= e^t (t^2 - 2t + 2) + C_2 \\
&= e^{\sqrt[3]{3x+4}} \left((3x+4)^{\frac{2}{3}} - 2(3x+4)^{\frac{1}{3}} + 2 \right) + C_2
\end{aligned}$$

6a. $u = e^x \Rightarrow \frac{du}{dx} = e^x = u$. Then:

$$\begin{aligned}
\int \frac{1}{e^x + 2e^{-x}} dx &= \int \frac{1}{u + \frac{2}{u}} \left(\frac{du}{u} \right) \\
&= \int \frac{1}{u^2 + 2} du \\
&= \int \frac{1}{(\sqrt{2})^2 + u^2} du \\
&= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + c \\
&= \frac{\sqrt{2}}{2} \tan^{-1} \left(\frac{e^x}{\sqrt{2}} \right) + c
\end{aligned}$$

$$\begin{aligned}
6b. \quad & \int_0^1 \frac{4x-5}{\sqrt{3+2x-x^2}} dx \\
&= \int_0^1 \frac{-2(2-2x)-1}{\sqrt{3+2x-x^2}} dx \\
&= -2 \int_0^1 \frac{2-2x}{\sqrt{3+2x-x^2}} dx - \int_0^1 \frac{1}{\sqrt{3+2x-x^2}} dx \\
&= -2 \int_0^1 \frac{2-2x}{\sqrt{3+2x-x^2}} dx - \int_0^1 \frac{1}{\sqrt{4-(x-1)^2}} dx \\
&= -2 \left[2\sqrt{3+2x-x^2} \right]_0^1 - \left[\sin^{-1} \left(\frac{x-1}{2} \right) \right]_0^1 \\
&= -2 \left[2\sqrt{4}-2\sqrt{3} \right]_0^1 - \left[\sin^{-1} \left(\frac{1-1}{2} \right) - \sin^{-1} \left(\frac{0-1}{2} \right) \right] \\
&= 4\sqrt{3} - 8 - 0 - \frac{\pi}{6} \\
&= \frac{24\sqrt{3}-48-\pi}{6}
\end{aligned}$$

$$\begin{aligned}
7 \text{ (a)} \quad & \int \frac{1}{3-4x-4x^2} dx \\
&= \int \frac{1}{4-(2x+1)^2} dx \\
&= \frac{1}{2} \int \frac{2}{4-(2x+1)^2} dx \\
&= \frac{1}{2} \cdot \frac{1}{2(2)} \ln \left| \frac{2+(2x+1)}{2-(2x+1)} \right| + c \\
&= \frac{1}{8} \ln \left| \frac{3+2x}{1-2x} \right| + c
\end{aligned}$$

Alternative Soln:

$$\begin{aligned}
& \int \frac{1}{3-4x-4x^2} dx \\
&= - \int \frac{1}{4x^2+4x-3} dx \\
&= - \int \frac{1}{(3+2x)(2x-1)} dx \\
&= - \frac{1}{4} \int \left(\frac{1}{2x-1} - \frac{1}{2x+3} \right) dx \\
&= - \frac{1}{4} \left[\frac{1}{2} \ln |2x-1| - \frac{1}{2} \ln |2x+3| \right] \\
&= - \frac{1}{8} \ln \left| \frac{2x-1}{2x+3} \right| + c
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad & \int_0^{\ln 3} \frac{e^{2x}}{\sqrt{1+e^x}} dx \\
&= \int_{\sqrt{2}}^2 \frac{(u^2-1)^2}{u} \cdot \frac{dx}{du} du \\
&= \int_{\sqrt{2}}^2 \frac{(u^2-1)^2}{u} \cdot \frac{2u}{u^2-1} du \\
&= 2 \int_{\sqrt{2}}^2 (u^2-1) du \\
&= 2 \left[\frac{u^3}{3} - u \right]_{\sqrt{2}}^2 \\
&= 2 \left[\left(\frac{8}{3} - 2 \right) - \left(\frac{2\sqrt{2}}{3} - \sqrt{2} \right) \right] \\
&= \frac{2}{3} (2 + \sqrt{2})
\end{aligned}$$

$$\text{Let } u^2 = 1 + e^x,$$

$$2u \frac{du}{dx} = e^x = u^2 - 1$$

$$\therefore \frac{du}{dx} = \frac{u^2-1}{2u} \Rightarrow \frac{dx}{du} = \frac{2u}{u^2-1}$$

8.

$$u = 3x - 1 \Rightarrow \frac{du}{dx} = 3$$

$$\text{When } x = \frac{2}{3}, u = 1$$

$$\text{When } x = 1, u = 2$$

$$\begin{aligned}
\int_{\frac{2}{3}}^1 \left(\frac{x^2}{3x-1} \right) dx &= \int_1^2 \frac{(u+1)^2}{\frac{9}{u}} \frac{1}{3} du \\
&= \frac{1}{27} \int_1^2 \frac{(u+1)^2}{u} du \\
&= \frac{1}{27} \int_1^2 \frac{u^2 + 2u + 1}{u} du \\
&= \frac{1}{27} \int_1^2 \left(u + 2 + \frac{1}{u} \right) du \\
&= \frac{1}{27} \left[(2 + 4 + \ln 2) - \left(\frac{1}{2} + 2 + \ln 1 \right) \right] \\
&= \frac{1}{27} \left(\frac{7}{2} + \ln 2 \right)
\end{aligned}$$

$$9 \quad \frac{d}{dx} \left(\tan \left(\frac{x}{2} \right) \right) = \frac{1}{2} \sec^2 \left(\frac{x}{2} \right) = \frac{1}{2 \cos^2 \left(\frac{x}{2} \right)} = \frac{1}{1 + \cos x}.$$

Using integration by parts,

$$\begin{aligned} \int \left(\frac{x + \sin x}{1 + \cos x} \right) dx &= (x + \sin x) \tan \left(\frac{x}{2} \right) - \int \tan \left(\frac{x}{2} \right) (1 + \cos x) dx \\ &= (x + \sin x) \tan \left(\frac{x}{2} \right) - \int \tan \left(\frac{x}{2} \right) (2 \cos^2 \left(\frac{x}{2} \right)) dx \\ &= x \tan \left(\frac{x}{2} \right) + \sin x \tan \left(\frac{x}{2} \right) - \int \sin x dx \\ &= x \tan \left(\frac{x}{2} \right) + 2 \sin \left(\frac{x}{2} \right) \cos \left(\frac{x}{2} \right) \tan \left(\frac{x}{2} \right) + \cos x + A \\ &= x \tan \left(\frac{x}{2} \right) + 2 \sin^2 \left(\frac{x}{2} \right) + 1 - 2 \sin^2 \left(\frac{x}{2} \right) + A \\ &= x \tan \left(\frac{x}{2} \right) + C, \quad \text{where } C = 1 + A \end{aligned}$$

$$10 \quad (a) \quad \frac{d}{dx} (\cos x^3) = -3x^2 (\sin x^3)$$

$$\begin{aligned} &\int x^5 \sin x^3 dx \\ &= \int x^3 (x^2 \sin x^3) dx \\ &= x^3 \left(-\frac{1}{3} \cos x^3 \right) - \int \left(-\frac{1}{3} \cos x^3 \right) 3x^2 dx \\ &= -\frac{x^3}{3} \cos x^3 + \int x^2 \cos x^3 dx \\ &= -\frac{x^3}{3} \cos x^3 + \frac{1}{3} \sin x^3 + c \end{aligned}$$

$$10 \quad (b) \quad x = \sqrt{5} \sec \theta \quad \text{when } x = \sqrt{10}, \theta = \frac{\pi}{4}$$

$$\frac{dx}{d\theta} = \sqrt{5} \sec \theta \tan \theta \quad \text{when } x = 2\sqrt{5}, \theta = \frac{\pi}{3}$$

$$\begin{aligned}
& \int_{\sqrt{10}}^{2\sqrt{5}} (x^2 - 5)^{-\frac{3}{2}} dx \\
&= \int_{\pi/4}^{\pi/3} (5 \sec^2 \theta - 5)^{-\frac{3}{2}} \sqrt{5} \sec \theta \tan \theta d\theta \\
&= \frac{1}{5} \int_{\pi/4}^{\pi/3} \frac{\sec \theta}{\tan^2 \theta} d\theta \\
&= \frac{1}{5} \int_{\pi/4}^{\pi/3} \frac{\cos \theta}{\sin^2 \theta} d\theta \quad \text{or} \quad \frac{1}{5} \int_{\pi/4}^{\pi/3} \cot \theta \operatorname{cosec} \theta d\theta \\
&= \frac{1}{5} \left[-\frac{1}{\sin \theta} \right]_{\pi/4}^{\pi/3} = \frac{1}{5} [-\operatorname{cosec} \theta]_{\pi/4}^{\pi/3} \\
&= \frac{1}{5} \sqrt{2} - \frac{2}{15} \sqrt{3} \quad \text{i.e.} \quad a = \frac{1}{5} \quad b = -\frac{2}{15}
\end{aligned}$$