Integration & Applications

1. MJC/I/2

(i) $\frac{d}{dx}\left(e^{x^2+1}\right) = 2xe^{x^2+1}$

(ii) $\int x^3 e^{x^2+1} dx$

$= \frac{1}{2}\int \left(2xe^{x^2+1}\right)\left(x^2\right) dx$

$= \frac{1}{2}\left(x^2 e^{x^2+1} - \int 2xe^{x^2+1} dx\right)$

$= \frac{1}{2}\left(x^2 e^{x^2+1} - e^{x^2+1}\right) + c = \frac{e^{x^2+1}}{2}\left(x^2 - 1\right) + c$

(iii) $\int_0^1 \left(x^3 e^{x^2+1} + e^2\right) dx$

$= \left[\frac{1}{2}\left[x^2 e^{x^2+1} - e^{x^2+1}\right] + e^2 x\right]_0^1$

$= \left[\frac{1}{2}\left(e^2 - e^2\right) + e^2\right] - \left[\frac{1}{2}(-e)\right]$

$= e^2 + 0.5e$

2. ACJC/II/4

(a) $\frac{d}{dx}\left(\cos x^3\right) = -3x^2\left(\sin x^3\right)$

$\int x^5 \sin x^3 \, dx$

$= \int x^3 (x^2 \sin x^3) \, dx$

$= x^3\left(-\frac{1}{3}\cos x^3\right) - \int\left(-\frac{1}{3}\cos x^3\right) 3x^2 \, dx$

$= -\frac{x^3}{3}\cos x^3 + \int x^2 \cos x^3 \, dx$

$= -\frac{x^3}{3}\cos x^3 + \frac{1}{3}\sin x^3 + c$

(b) $x = \sqrt{5}\sec\theta$ $\qquad$ *when* $x = \sqrt{10},\ \theta = \frac{\pi}{4}$

$\frac{dx}{d\theta} = \sqrt{5}\sec\theta\tan\theta$ $\qquad$ *when* $x = 2\sqrt{5},\ \theta = \frac{\pi}{3}$

$$\int_{\sqrt{10}}^{2\sqrt{5}} \left(x^2-5\right)^{-\frac{3}{2}}\, dx$$

$$= \int_{\pi/4}^{\pi/3} \left(5\sec^2\theta - 5\right)^{-\frac{3}{2}} \sqrt{5}\sec\theta\tan\theta\, d\theta$$

$$= \frac{1}{5}\int_{\pi/4}^{\pi/3} \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \frac{1}{5}\int_{\pi/4}^{\pi/3} \frac{\cos\theta}{\sin^2\theta}\, d\theta \quad or \quad \frac{1}{5}\int_{\pi/4}^{\pi/3} \cot\theta \operatorname{cosec}\theta\, d\theta$$

$$= \frac{1}{5}\left[-\frac{1}{\sin\theta}\right]_{\pi/4}^{\pi/3} \qquad = \frac{1}{5}\left[-\operatorname{cosec}\theta\right]_{\pi/4}^{\pi/3}$$

$$= \frac{1}{5}\sqrt{2} - \frac{2}{15}\sqrt{3} \qquad i.e.\ a = \frac{1}{5} \qquad b = -\frac{2}{15}$$

3. DHS/I/6

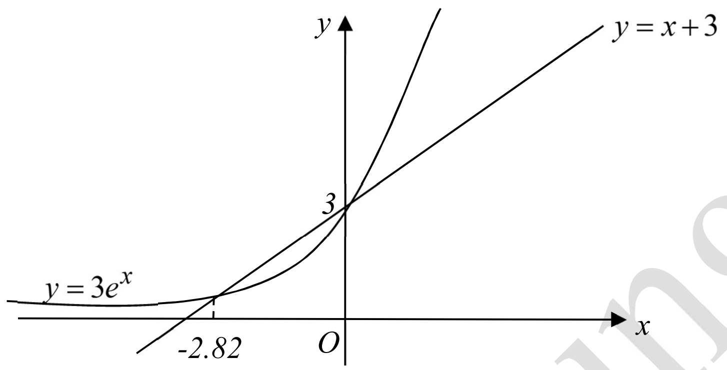
6(a)	$\int \frac{e^t}{(1+3e^t)^2}\, dt = \frac{1}{3}\int 3e^t(1+3e^t)^{-2}\, dt$ $= \frac{(1+3e^t)^{-1}}{-3} + c = -\frac{1}{3(1+3e^t)} + c$
6(b)	$\frac{d}{dx}\left(\tan\left(x^2\right)\right) = 2x\sec^2\left(x^2\right)$ $\int x^3\sec^2\left(x^2\right) dx = \frac{1}{2}\int x^2\left[2x\sec^2\left(x^2\right)\right] dx$ $= \frac{1}{2}\left[x^2\tan\left(x^2\right) - \int 2x\tan\left(x^2\right) dx\right]$ $= \frac{1}{2}\left[x^2\tan\left(x^2\right) - \ln\left\|\sec\left(x^2\right)\right\|\right] + c$
6(c)	$\int_0^4 x^2\|x-3\|\, dx$ $= -\int_0^3 x^2(x-3)\, dx + \int_3^4 x^2(x-3)\, dx$ $= -\left[\frac{x^4}{4} - x^3\right]_0^3 + \left[\frac{x^4}{4} - x^3\right]_3^4$ $= \frac{27}{2}$ or 13.5

4. JJC/I/10

(i)

$$\int \frac{x-1}{1+4x^2}\,dx$$
$$=\frac{1}{8}\int \frac{8x}{1+4x^2}\,dx-\int \frac{1}{1+4x^2}\,dx$$
$$=\frac{1}{8}\ln\left(1+4x^2\right)-\frac{1}{2}\tan^{-1}(2x)+c$$

(ii)



From the graphs, $x<-2.82$ or $x>0$.

For $x<-2.82$ or $x>0$, $3e^x>x+3$

i.e. $3e^x-x-3>0$ if $x<-2.82$ or $x>0$

For $-2.82<x<0$, $3e^x<x+3$

i.e. $3e^x-x-3<0$ if $-2.82<x<0$

$$\int_{-2}^{2}\left|3e^x-x-3\right|dx$$
$$=\int_{-2}^{0}-(3e^x-x-3)dx+\int_{0}^{2}(3e^x-x-3)dx$$
$$=-[3e^x-\frac{x^2}{2}-3x]_{-2}^{0}+[3e^x-\frac{x^2}{2}-3x]_{0}^{2}$$
$$=-[3-(3e^{-2}-2+6)]+[(3e^2-2-6)-3]$$
$$=3e^{-2}+3e^2-10$$

5. ACJC/II/1

$$\int_{-1}^{1} \left| e^{2x} - \frac{1}{e^{2(x-1)}} \right| \, dx$$

$$= -\int_{-1}^{1/2} e^{2x} - \frac{1}{e^{2(x-1)}} \, dx + \int_{1/2}^{1} e^{2x} - \frac{1}{e^{2(x-1)}} \, dx$$

$$= -\left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2(x-1)} \right]_{-1}^{1/2} + \left[\frac{1}{2}e^{2x} + \frac{1}{2}e^{-2(x-1)} \right]_{1/2}^{1}$$

$$= \frac{1}{2}\left(e^{4} + e^{2} - 4e + e^{-2} + 1\right)$$

6. MI/I/11

(a) Let $x = \cos\theta \Rightarrow \frac{dx}{d\theta} = -\sin\theta$

When $\theta = \frac{\pi}{2}$, $x = 0$. When $\theta = \pi$, $x = -1$.

$$\int_{\pi/2}^{\pi} \frac{\sin\theta}{1+\cos^2\theta} \, d\theta = \int_{0}^{-1} \frac{-1}{1+x^2} \, dx$$

$$= -\left[\tan^{-1} x\right]_0^{-1}$$

$$= \frac{\pi}{4}$$

(b)(i)

$$\int e^x \sin 2x \, dx = e^x \sin 2x - \int e^x . 2\cos 2x \, dx$$

$$= e^x \sin 2x - 2\left[e^x \cos 2x + \int e^x . 2\sin 2x \, dx \right]$$

$$5\int e^x \sin 2x \, dx = e^x \sin 2x - 2e^x \cos 2x + C$$

$$\therefore \int e^x \sin 2x \, dx = \frac{1}{5} e^x \left[\sin 2x - 2\cos 2x\right] + C$$

(b)(ii)

$$5\int_0^{\pi/2} \left(e^x \sin 2x\right) dx = 5 \times \frac{1}{5}\left[e^x\left(\sin 2x - 2\cos 2x\right)\right]_0^{\pi/2}$$

$$= 2\left(e^{\pi/2} + 1\right)$$

7. SAJC/I/11

11(a)	Using $A(\sin\theta + \cos\theta) + B(\cos\theta - \sin\theta) \equiv 4\sin\theta$, Comparing the coefficient of $\sin\theta$ & $\cos\theta$ respectively, We have $A - B = 4$ ----(1) $A + B = 0$ --- (2) Solving the simultaneous equations, $A = 2$, $B = -2$ (Ans) $\int_0^{\frac{1}{4}\pi} \frac{4\sin\theta}{\sin\theta+\cos\theta}\,\mathrm{d}\theta$ $= \int_0^{\frac{1}{4}\pi} \frac{2(\sin\theta+\cos\theta)-2(\cos\theta-\sin\theta)}{\sin\theta+\cos\theta}\,\mathrm{d}\theta$ $= \int_0^{\frac{1}{4}\pi} 2\,\mathrm{d}\theta - 2\int_0^{\frac{1}{4}\pi} \frac{(\cos\theta-\sin\theta)}{\sin\theta+\cos\theta}\,\mathrm{d}\theta$ $= 2[\theta]_0^{\frac{\pi}{4}} - 2[\ln(\sin\theta+\cos\theta)]_0^{\frac{\pi}{4}}$ $= \frac{\pi}{2} - \ln 2$ (or $\frac{\pi}{2} - 2\ln\sqrt{2}$)
(b)(i)	$\frac{2t}{(t+1)^2} = \frac{A}{(t+1)} + \frac{B}{(t+1)^2}$ Solving, we have $A = 2$, $B = -2$
(ii)	Using substitution $t = \sqrt{2x-1}$, $x = \frac{t^2+1}{2}$, $\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{\sqrt{2x-1}} = \frac{1}{t}$ When $x = 1$, $t = 1$ $x = 5$, $t = 3$ Hence $\int_1^5 \frac{1}{x+\sqrt{2x-1}}\,\mathrm{d}x = \int_1^3 \frac{1}{\left(\frac{t^2+1}{2}\right)+t}(t)\,\mathrm{d}t$ $= \int_1^3 \frac{2t}{t^2+2t+1}\,\mathrm{d}t$ $= \int_1^3 \frac{2t}{(t+1)^2}\,\mathrm{d}t$ $= 2\int_1^3 \left(\frac{1}{t+1} - \frac{1}{(t+1)^2}\right)\mathrm{d}t$ from **b(i)** $= 2[\ln(t+1) + \frac{1}{t+1}]_1^3$ $= 2\ln 2 - \frac{1}{2}$

8. SRJC/I/2

(i) Now, $-\frac{\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$ where $|x| \le 1$

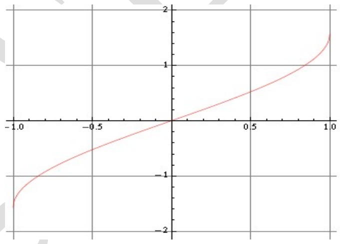
Let $y = \sin^{-1} x \Rightarrow \sin y = x \Rightarrow \cos y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\pm\sqrt{1-\sin^2 y}}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\pm\sqrt{1-x^2}}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{+\sqrt{1-x^2}}$$



(since from the graph $\frac{\mathrm{d}y}{\mathrm{d}x} \ge 0$ for all $|x| \le 1$)

$$\left[\sin^{-1} x\right]_0^{\frac{1}{\sqrt{2}}} = \int_a^{\frac{\pi}{4}} (\cos 2x + 1)\,\mathrm{d}x$$

$$\frac{\pi}{4} - 0 = \left[\frac{1}{2}\sin 2x + x\right]_a^{\frac{\pi}{4}}$$

$$\frac{\pi}{4} = \left(\frac{1}{2} + \frac{\pi}{4}\right) - \left(\frac{1}{2}\sin 2a + a\right) \qquad \Rightarrow \sin 2a + 2a - 1 = 0$$

9. AJC/I/12

(i) $\displaystyle\int_1^{\infty} \frac{t^5}{(1+t^3)^3}\,dt = \int_1^0 \frac{\frac{1}{u^5}}{(1+\frac{1}{u^3})^3}\left(-\frac{1}{u^2}\right)du = \int_0^1 \frac{u^2}{(u^3+1)^3}du$ (shown)

$$= \frac{1}{3}\int_0^1 3u^2(1+u^3)^{-3}du = \frac{1}{3}\left[\frac{(1+u^3)^{-2}}{-2}\right]_0^1 \qquad = -\frac{1}{24} + \frac{1}{6} = \frac{1}{8}$$

(ii) As $t \to \infty$, x, $y \to 0$. The point is $(0, 0)$

(iii) Area required $= \displaystyle\int_0^{\frac{1}{2}} y\,dx = \int_{\infty}^1 \frac{t^2}{1+t^3} \bullet \frac{(1-2t^3)}{(1+t^3)^2}dt \quad = \int_1^{\infty} \frac{2t^5 - t^2}{(1+t^3)^3}dt$

$$= \int_1^{\infty} \frac{2t^5}{(1+t^3)^3}dt - \int_1^{\infty} \frac{t^2}{(1+t^3)^3}dt$$

$$= 2\left(\frac{1}{8}\right) - \frac{1}{3}\int_1^{\infty} 3t^2(1+t^3)^{-3}dt$$

$$= \frac{1}{4}-\frac{1}{3}\left[\frac{(1+t^3)^{-2}}{-2}\right]_1^{\infty} \quad = \frac{1}{4}+\frac{1}{6}\left(0-\frac{1}{4}\right)=\frac{5}{24} \text{ units}^2$$

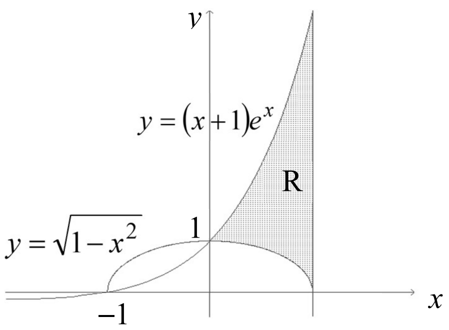
10. IJC/I/11

11a i	$2\cos 3x\cos x=\cos(4x)+\cos(2x)$ $\int \cos 3x\cos x \ \mathrm{d}x=\frac{1}{2}\int 2\cos 3x\cos x \ \mathrm{d}x$ $=\frac{1}{2}\int(\cos 4x+\cos 2x) \ \mathrm{d}x$ $=\frac{1}{2}\left(\frac{1}{4}\sin 4x+\frac{1}{2}\sin 2x\right)+c$ $=\frac{1}{8}\sin 4x+\frac{1}{4}\sin 2x+c$
ii	$\int_0^{\frac{\pi}{4}} x\cos 3x\cos x \ \mathrm{d}x$ $=x\left[\frac{1}{8}\sin 4x+\frac{1}{4}\sin 2x\right]_0^{\frac{\pi}{4}}-\int_0^{\frac{\pi}{4}}\left(\frac{1}{8}\sin 4x+\frac{1}{4}\sin 2x\right) \mathrm{d}x$ $=\left(\frac{\pi}{4}\right)\left[\frac{1}{8}\sin 4\left(\frac{\pi}{4}\right)+\frac{1}{4}\sin 2\left(\frac{\pi}{4}\right)\right]-\left[-\frac{1}{32}\cos 4x-\frac{1}{8}\cos 2x\right]_0^{\frac{\pi}{4}}$ $=\left(\frac{\pi}{4}\right)\left[\frac{1}{8}\sin\pi+\frac{1}{4}\sin\left(\frac{\pi}{2}\right)\right]+\left[\frac{1}{32}\cos 4\left(\frac{\pi}{4}\right)+\frac{1}{8}\cos 2\left(\frac{\pi}{4}\right)-\left(\frac{1}{32}+\frac{1}{8}\right)\right]$ $=\frac{\pi}{16}+\left[-\frac{1}{32}-\frac{1}{32}-\frac{1}{8}\right]=\frac{\pi-3}{16}$
11b	Method 1: (Translation – Stretching) 1: Translation by $\frac{3}{2}\pi$ in the direction of the x-axis. 2: Stretching parallel to the x-axis with scale factor of $\frac{1}{2}$. Method 2: (Stretching – Translation) 1: Stretching parallel to the x-axis with scale factor of $\frac{1}{2}$. 2: Translation by $\frac{3}{4}\pi$ in the direction of the x-axis.

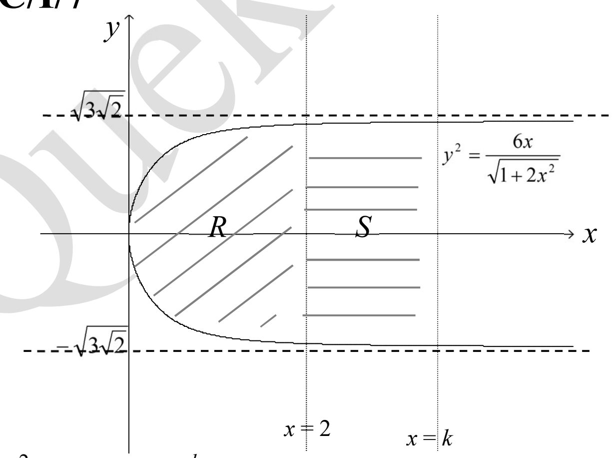
11c	Area bounded by the curves $= \int_0^{1.0471976} \left[\sin\left(2x - \frac{3}{2}\pi\right) - \cos 3x \cos x \right] dx$ = 0.32476 = <u>0.325 units^2 (to 3 sf)</u>

11. YJC/I/6

6 (i)	Let $x = a\sin\theta$, then $\frac{dx}{d\theta} = a\cos\theta$ When $x = 0$, $\theta = 0$ When $x = a$, $\theta = \frac{\pi}{2}$ $\int_0^a \sqrt{a^2 - x^2}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{a^2 - a^2\sin^2\theta}\,(a\cos\theta) d\theta$ $= \int_0^{\frac{\pi}{2}} a\sqrt{\cos^2\theta}\,(a\cos\theta) d\theta$ $= \int_0^{\frac{\pi}{2}} a^2\cos^2\theta\, d\theta$ $= a^2 \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right) d\theta$ $= a^2 \left[\frac{1}{4}\sin 2\theta + \frac{1}{2}\theta\right]_0^{\frac{\pi}{2}}$ $= a^2\left[\left(\frac{1}{4}\sin\pi + \frac{\pi}{4}\right) - \left(\frac{1}{4}\sin 0 + 0\right)\right]$ $= \frac{\pi}{4}a^2$ //
(ii)	Let $u = x + n$, $\frac{dv}{dx} = e^{nx}$ $\frac{du}{dx} = 1$, $v = \frac{1}{n}e^{nx}$ $\int_0^n (x+n)e^{nx}\, dx$

	$=\left[(x+n)(\frac{1}{n}e^{nx})\right]_0^n-\int_0^n \frac{1}{n}e^{nx}\,\mathrm{d}x$ $=\left[(2n)(\frac{1}{n}e^{n^2})-(n)(\frac{1}{n}e^{0})\right]-\left[\frac{1}{n^2}e^{nx}\right]_0^n$ $=\left[2e^{n^2}-1\right]-\left[\frac{1}{n^2}e^{n^2}-\frac{1}{n^2}e^{0}\right]$ $=2e^{n^2}-1-\frac{1}{n^2}e^{n^2}+\frac{1}{n^2}$ //
(iii)	 Area of R = $\int_0^1 (1+x)e^x\,\mathrm{d}x - \int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$ $=(2e^1-1-e^1+1)-(\frac{\pi}{4})$ $=e-\frac{\pi}{4}$ units^2 //

12. YJC/I/7



Given: $\pi\int_0^2 y^2\,\mathrm{d}x = \pi\int_2^k y^2\,\mathrm{d}x$

$$\pi\int_0^2 \frac{6x}{\sqrt{1+2x^2}}\,\mathrm{d}x = \pi\int_2^k \frac{6x}{\sqrt{1+2x^2}}\,\mathrm{d}x$$

$$\frac{6}{4}\int_0^2 \frac{4x}{\sqrt{1+2x^2}}\,\mathrm{d}x = \frac{6}{4}\int_2^k \frac{4x}{\sqrt{1+2x^2}}\,\mathrm{d}x$$

$$\left[2\sqrt{1+2x^2}\right]_0^2 = \left[2\sqrt{1+2x^2}\right]_2^k$$

$$2(3)-2 = 2\sqrt{1+2k^2}-2(3)$$

$$\sqrt{1+2k^2} = 5$$

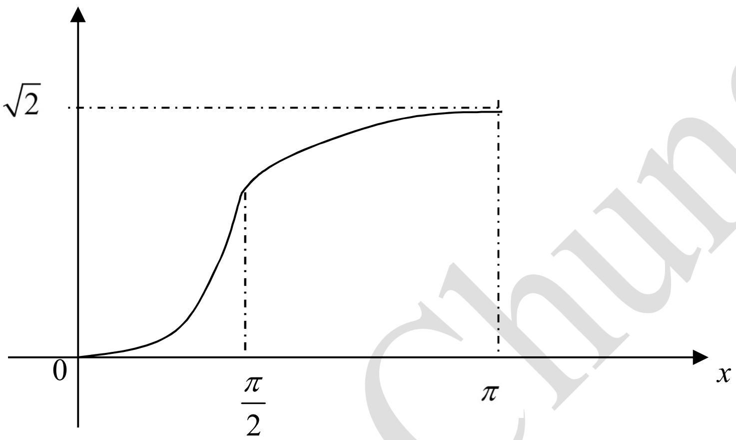
$$1+2k^2 = 25$$

$$k^2 = 12$$

$$k = \sqrt{12} \qquad (\because k > 2)\ //$$

13. AJC/I/2

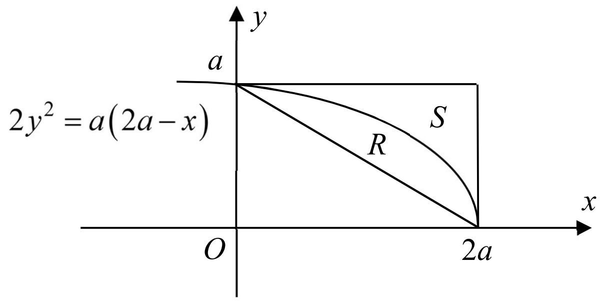
(i)



(ii) Volume required $= \pi(\sqrt{2})^2(\pi) - \pi\int_0^{\frac{\pi}{2}} \tan^2\frac{x}{2}\,dx - \pi\int_{\frac{\pi}{2}}^{\pi} \frac{2}{\pi}x\,dx$

$$= 2\pi^2 - \pi\left[2\tan\frac{x}{2} - x\right]_0^{\frac{\pi}{2}} - \left[x^2\right]_{\frac{\pi}{2}}^{\pi} \qquad = \left(\frac{7}{4}\pi^2 - 2\pi\right) \text{units}^3$$

14. ACJC/I/11

11 (i)	
(ii)	When S is rotated completely about the x-axis, Required volume $= \pi a^2(2a) - \pi\int_0^{2a} \frac{a}{2}(2a-x)\,dx$ $= 2\pi a^3 - \frac{\pi a}{2}\left[\frac{(2a-x)^2}{-2}\right]_0^{2a}$ $= 2\pi a^3 - \frac{\pi a}{2}(2a^2)$ $= \pi a^3$ cu. units
(iii)	After a translation of $2a$ units in the negative x-direction, New equation is $2y^2 = a(2a-(x+2a)) \Rightarrow x = -\frac{2y^2}{a}$ When R is rotated completely about the line $x = 2a$, Required volume $= \frac{1}{3}\pi(2a)^2(a) - \pi\int_0^a \left(-\frac{2y^2}{a}\right)^2 dy$ $= \frac{4}{3}\pi a^3 - \pi\left[\frac{4y^5}{5a^2}\right]_0^a$ $= \frac{4}{3}\pi a^3 - \frac{4}{5}\pi a^3 = \frac{8}{15}\pi a^3$ cu. units

15. CJC/I/10

(a) $\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\sin x - \int \frac{1}{2}e^{2x}\cos x\,dx$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\left[\frac{1}{2}e^{2x}\cos x + \int \frac{1}{2}e^{2x}\sin x\,dx\right]$$

$$= \frac{1}{2}e^{2x}\sin x - \frac{1}{4}e^{2x}\cos x - \frac{1}{4}\int e^{2x}\sin x\,dx$$

$$\Rightarrow \left(1+\frac{1}{4}\right)\int e^{2x}\sin x\,dx = \frac{1}{2}e^{2x}\left(\sin x - \frac{1}{2}\cos x\right) + C$$

$$\Rightarrow \int e^{2x}\sin x\,dx = \frac{2}{5}e^{2x}\left(\sin x - \frac{1}{2}\cos x\right) + D$$

(b) When $t = -\pi$, $x = -2\pi$
When $t = \pi$, $x = 2\pi$

Area bounded by C and the x-axis $= \int_{-\pi}^{\pi} 2(1-\cos t)[2(1-\cos t)]\,dt$

$$= 4\int_{-\pi}^{\pi}\left(1 - 2\cos t + \cos^2 t\right)dt$$

$$= 4\int_{-\pi}^{\pi}\left(1 - 2\cos t + \frac{\cos 2t + 1}{2}\right)dt$$

$$= 4\left[t - 2\sin t + \frac{1}{2}\left(\frac{1}{2}\sin 2t + t\right)\right]_{-\pi}^{\pi}$$

$$= 4\left[\pi + \frac{1}{2}\pi - \left(-\pi - \frac{1}{2}\pi\right)\right]$$

$$= 12\pi$$

Area required $= 4(2)(2\pi) - 12\pi$
$= 4\pi$ sq units

(c) Point of intersection : (-1.15995, 3.84005)

Volume of solid

$$= \pi\int_{1}^{3.84005} \ln y\,dy + \frac{1}{3}\pi(1.15995)^2(5 - 3.84005)$$

= 8.94 cubic units

16. NYJC/I/12

(a)(i) $\frac{d}{dx}(1-x^2)^n = -2xn(1-x^2)^{n-1}$

(ii) $\int x^3(1-x^2)^n\,dx = -\frac{x^2(1-x^2)^{n+1}}{2(n+1)} - \int -2x\cdot\frac{1}{2(n+1)}(1-x^2)^{n+1}dx$

$$= -\frac{x^2(1-x^2)^{n+1}}{2(n+1)} + \frac{1}{n+1}\int x\cdot(1-x^2)^{n+1}\mathrm{d}x$$

$$= -\frac{x^2(1-x^2)^{n+1}}{2(n+1)} - \frac{1}{2}\frac{1}{(n+1)(n+2)}(1-x^2)^{n+2} + C$$

(b)(i) Area of A =

$$\sqrt{2}\left(\frac{3\pi}{4}-\frac{\pi}{4}\right) - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin x + \cos x \, dx$$

$$= \frac{\pi\sqrt{2}}{2} - \left[\sin x - \cos x\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \frac{\pi\sqrt{2}}{2} - \sqrt{2}$$

(ii) By translation, req'd area = $\frac{\pi\sqrt{2}}{2}$ – Area of $A = \sqrt{2}$

(iii) Volume = $\pi(\sqrt{2})^2\frac{\pi}{2} - \pi\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(\sin x + \cos x)^2 \, dx$

$= 4.93$

17. PJC/I/12

(a) $\mathrm{d}u = e^x\mathrm{d}x \Rightarrow \mathrm{d}x = \frac{\mathrm{d}u}{e^x} = \frac{\mathrm{d}u}{u-4}$

When $x = 0, u = 5$, When $x = \ln p$, $u = 4 + p$

$$\therefore \int_0^{\ln p} \frac{1}{4+e^x}\,\mathrm{d}x = \int_5^{4+p} \frac{1}{u}\frac{\mathrm{d}u}{u-4}$$

$$= \frac{1}{4}\int_5^{4+p}\left(\frac{1}{u-4} - \frac{1}{u}\right)\mathrm{d}u \quad \text{(by partial frac)}$$

$$= \frac{1}{4}\Big[\ln|u-4| - \ln|u|\Big]_5^{4+p}$$

$$= \frac{1}{4}\Big[\big(\ln p - \ln(4+p)\big) - \big(\ln 1 - \ln 5\big)\Big]$$

(Note that since $p > 0$, implied by the qn, $4 + p > 0$. Hence modulus can be dropped.)

$$= \frac{1}{4}\ln\left(\frac{5p}{4+p}\right) \quad \text{(shown)}$$

(i)

Volume of revolution $= \pi \int_0^{\ln 6} \frac{1}{4+e^x}\,dx$

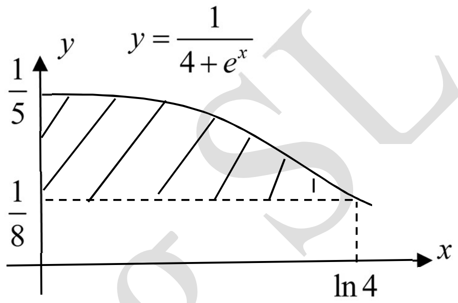
$= \frac{\pi}{4}\ln\left(\frac{5\times 6}{4+6}\right)$ (by earlier result)

$= \frac{\pi}{4}\ln 3$

(ii)

Let $x = \ln\left(\frac{1-4y}{y}\right) \Rightarrow e^x = \frac{1-4y}{y} \Rightarrow y = \frac{1}{4+e^x}$

When $y = \frac{1}{5}$, $x = 0$, When $y = \frac{1}{8}$, $x = \ln 4$



Since $\int_{1/8}^{1/5} \ln\left(\frac{1-4y}{y}\right) dy$ = area of shaded region

$= \int_0^{\ln 4} \frac{1}{4+e^x}\,dx - \frac{1}{8}\ln 4$

$= \frac{1}{4}\ln\left(\frac{5}{2}\right) - \frac{1}{4}\ln 2$

$= \frac{1}{4}\ln\left(\frac{5}{4}\right)$

18. DHS/I/4

4(i)	Area of $R = \int_{-1}^{0} y\,dx = \int_{-1}^{0}\left[x^3+1\right] dx = \frac{3}{4}$ $y = 1 + x^3 \Rightarrow x = (y-1)^{\frac{1}{3}}$ Area of $S = \int_2^b x\,dy = \int_2^b (y\text{-}1)^{\frac{1}{3}}\,dy = \frac{3}{4}\left[(y-1)^{\frac{4}{3}}\right]_2^b$ $= \frac{3}{4}\left[(b-1)^{\frac{4}{3}} - 1\right]$ Equating and solve for b: $\frac{3}{4}\left[(b-1)^{\frac{4}{3}} - 1\right] = \frac{3}{4}$ $\Rightarrow b = 1 + 2^{\frac{3}{4}} = 2.68$ (3 s.f.)
4(ii)	For $y=b$, $x = (b-1)^{\frac{1}{3}} = 1.1892 = k$ (say) Volume required

	$= \pi\left[b^2k - 2^2(1) - \int_1^k (x^3+1)^2\,\mathrm{d}x\right]$ $= 3.53\pi$ (or 11.1) (unit cube)

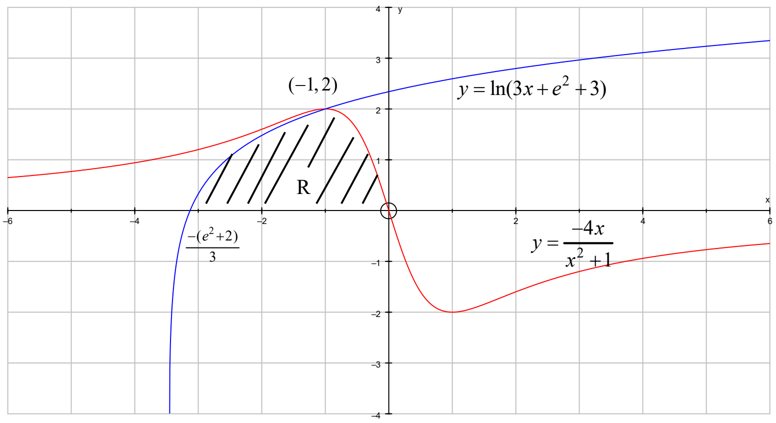
19. MJC/II/5

(i)	Area of R = Area of quadrant-Area of triangle
	$=\int_{-a}^{0}\sqrt{b^2-\dfrac{b^2x^2}{a^2}}\,\mathrm{d}x-\dfrac{1}{2}ab$ y
	$=-\dfrac{1}{2}ab+\int_{-a}^{0}\dfrac{b}{a}\sqrt{a^2-x^2}\,\mathrm{d}x$ (shown)
	$x=a\sin\theta$
	$\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=a\cos\theta$
	$\begin{pmatrix}0=a\sin\theta\Rightarrow\theta=0\\ -a=a\sin\theta\Rightarrow\theta=-\dfrac{\pi}{2}\end{pmatrix}$
	$\int_{-a}^{0}\dfrac{b}{a}\sqrt{a^2-x^2}\,\mathrm{d}x-\dfrac{1}{2}ab$
	$=\dfrac{b}{a}\int_{-\frac{\pi}{2}}^{0}\sqrt{a^2-(a\sin\theta)^2}\,(a\cos\theta)\,\mathrm{d}\theta-\dfrac{1}{2}ab$
	$=\dfrac{a^2b}{a}\int_{-\frac{\pi}{2}}^{0}\sqrt{1-\sin^2\theta}\,(\cos\theta)\,\mathrm{d}\theta-\dfrac{1}{2}ab$
	$=ab\int_{-\frac{\pi}{2}}^{0}\cos^2\theta\,\mathrm{d}\theta-\dfrac{1}{2}ab$
	$=ab\int_{-\frac{\pi}{2}}^{0}\dfrac{1+\cos 2\theta}{2}\,\mathrm{d}\theta-\dfrac{1}{2}ab$
	$=ab\left[\dfrac{1}{2}\theta+\dfrac{\sin 2\theta}{4}\right]_{-\frac{\pi}{2}}^{0}-\dfrac{1}{2}ab$
	$=ab\left[0+0-\left(\dfrac{1}{2}\left(-\dfrac{\pi}{2}\right)+\dfrac{\sin(-\pi)}{4}\right)\right]-\dfrac{1}{2}ab$
	$=ab\left(\dfrac{\pi}{4}\right)-\dfrac{1}{2}ab$
	$=\dfrac{ab}{4}(\pi-2)\text{units}^2$

(ii)	Volume generated $=\pi\int_0^b x^2\,\mathrm{dy}-\frac{1}{3}\pi a^2 b$ $=\pi\int_0^b a^2-\frac{a^2y^2}{b^2}\,\mathrm{dy}-\frac{1}{3}\pi a^2 b$ $=\pi\left[a^2y-\frac{a^2y^3}{3b^2}\right]_0^b-\frac{1}{3}\pi a^2 b$ $=\pi\left[a^2b-\frac{a^2b^3}{3b^2}\right]-\frac{1}{3}\pi a^2 b$ $=\pi\left[a^2b-\frac{a^2b}{3}\right]-\frac{1}{3}\pi a^2 b$ $=\pi\left(\frac{2}{3}a^2b\right)-\frac{1}{3}\pi a^2 b$ $=\frac{1}{3}\pi a^2 b$

20. RI(JC)/II/4

Q4 [12] **(i)[2]**	$\int_{-1}^{1}\mathrm{f}(\lvert x\rvert)\mathrm{d}x=2\int_0^1\frac{-4x}{x^2+1}\mathrm{d}x$ $=2(-2)\int_0^1\frac{2x}{x^2+1}\mathrm{d}x$ $=-4\left[\ln(x^2+1)\right]_0^1$ $=-4\ln 2$
(ii)[7]	$\int\ln(3x+\mathrm{e}^2+3)\mathrm{d}x$ $=\int\ln(u)\frac{1}{3}\mathrm{d}u$ $=\frac{1}{3}\int\ln(u)\mathrm{d}u$ $=\frac{1}{3}[u\ln u]-\frac{1}{3}\int 1\mathrm{d}u$ $=\frac{1}{3}[u\ln u-u]+c$ $=\frac{1}{3}\left[(3x+\mathrm{e}^2+3)\ln(3x+\mathrm{e}^2+3)-(3x+\mathrm{e}^2+3)\right]+c$



$$\int_{-\frac{(e^2+2)}{3}}^{-1} \ln(3x+e^2+3)\,\mathrm{d}x + \int_{-1}^{0} \frac{-4x}{x^2+1}\,\mathrm{d}x$$

$$=\frac{1}{3}\Big[(3x+e^2+3)\ln(3x+e^2+3)-(3x+e^2+3)\Big]_{-\frac{(e^2+2)}{3}}^{-1}+2\ln 2$$

$$=\frac{1}{3}[e^2\ln e^2-e^2+1]+2\ln 2$$

$$=\frac{1}{3}(e^2+1)+2\ln 2$$

(iii)[3]

$$\pi\int_{-\frac{(e^2+2)}{3}}^{-1}\left(\ln(3x+e^2+3)\right)^2\mathrm{d}x+\pi\int_{-1}^{0}\left(\frac{-4x}{x^2+1}\right)^2\mathrm{d}x$$

$= 20.55$ (2d.p.)

21. SAJC/II/4

(i) Stationary points: Let $\frac{dy}{dx}=0 \Rightarrow \frac{\sqrt{8}(x^2+1)-\sqrt{8}x(2x)}{\left(x^2+1\right)^2}=0$

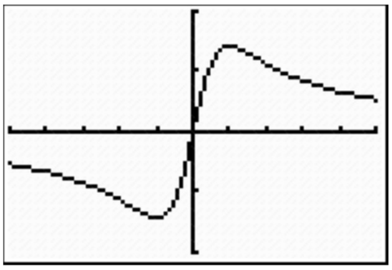
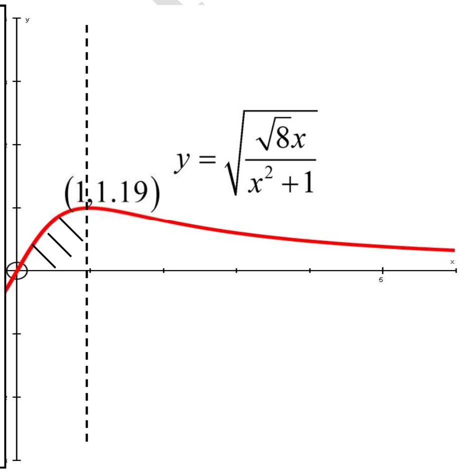
$\Rightarrow -x^2+1=0 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

$\Rightarrow y=\frac{\sqrt{8}}{1+1}=\sqrt{2}$ or $y=\frac{\sqrt{8}(-1)}{(-1)^2+1}=-\sqrt{2}$

The stationary points are at $\left(1,\sqrt{2}\right)$ and $\left(-1,-\sqrt{2}\right)$.

(ii) Axes Intercepts: When $x=0$, $y=0$ $\Rightarrow$ $(0,0)$ is on the curve.

Horizontal asymptote: as $x \to \pm\infty$, $y \to 0$ since y is a proper fraction, there is a horizontal asymptote at $y=0$

	
(iii)	$\int_0^n \mathrm{f}(x)\,\mathrm{d}x = \sqrt{2}\int_0^n \frac{2x}{x^2+1}\,\mathrm{d}u$ $= \sqrt{2}\left[\ln\left\|x^2+1\right\|\right]_0^n$ $= \sqrt{2}\left[\ln\left(n^2+1\right) - \ln\left(0^2+1\right)\right]$ $= \sqrt{2}\ln\left(n^2+1\right)$ (shown) $\int_{-2}^{2}\left\|\mathrm{f}(x)\right\|\,\mathrm{d}x$ $= 2\int_0^2 \mathrm{f}(x)\,\mathrm{d}x = 2\left[\sqrt{2}\ln\left(2^2+1\right)\right]$ $= 2\sqrt{2}\ln 5 = \sqrt{2}\ln 25$
(iv)	Volume obtained $= \pi\int_0^1 \left[g(x)\right]^2 dx$ $= \pi\int_0^1 f(x)\,dx$ $= \pi\sqrt{2}\ln\left[1^2+1\right]$ $=\pi\sqrt{2}\ln 2$ units^3 

22. NJC/I/2

2	$\int_0^p 3^x \ \mathrm{d}x = \frac{2}{\ln 3}$ $\left[\frac{3^x}{\ln 3}\right]_0^p = \frac{2}{\ln 3}$ $\frac{1}{\ln 3}\left(3^p - 3^0\right) = \frac{2}{\ln 3}$ $3^p - 1 = 2$ $3^p = 3$ $\therefore p = 1$

2(i)	Total area of all the *n* rectangles, *A* $= \frac{1}{n}\left(3^{\frac{1}{n}}\right) + \frac{1}{n}\left(3^{\frac{2}{n}}\right) + \frac{1}{n}\left(3^{\frac{3}{n}}\right) + \cdots + \frac{1}{n}\left(3^{\frac{n}{n}}\right)$ $= \frac{1}{n}\left(3^{\frac{1}{n}} + 3^{\frac{2}{n}} + 3^{\frac{3}{n}} + \cdots + 3^{\frac{n}{n}}\right)$ $= \frac{3^{\frac{1}{n}}}{n}\left(\frac{\left(3^{\frac{1}{n}}\right)^n - 1}{3^{\frac{1}{n}} - 1}\right)$ $= \frac{2\left(3^{\frac{1}{n}}\right)}{n\left(3^{\frac{1}{n}} - 1\right)}$
2(ii)	As $n \to \infty$, limit of *A* = area under the curve $y = 3^x$ for $0 \le x \le 1$ $= \int_0^1 3^x \ \mathrm{d}x$ $= \frac{2}{\ln 3}$