Vectors Revision Worksheet (Planes)

1) The plane Π contains the point B (3, 1, 0). Thus, two vectors parallel to plane Π will be $\begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$.

A normal vector of Π will be $\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} \times \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -12 \\ -2 \end{pmatrix}$.

Therefore, vector equation of plane Π is

$$\mathbf{r}.\begin{pmatrix} 3 \\ -12 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ -12 \\ -2 \end{pmatrix} = -3.$$

So, Cartesian equation of plane Π is $3x - 12y - 2z = -3$.

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2) Since O lies on the line l, thus the plane π contains OA and parallel to $\lambda(\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$. Thus a normal to the plane is

$$\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix}$$

$\therefore$ Eqn of the plane π is $\boldsymbol{r} \,.\, \begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix} = 0$

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3) Vector equation of plane is $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ -4 \end{pmatrix}$, where $s, t \in \mathbb{R}$

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4) l_1: $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$,

l_2: $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix} + \mu\begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix}, \mu \in \mathbb{R}$

Consider $\begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 8 \\ 11 \\ 6 \end{pmatrix} = \begin{pmatrix} 34 \\ -34 \\ 17 \end{pmatrix} = 17\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

A normal to Π_1 is $\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

$$\Pi_1 : \mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \mathbf{r} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 4$$

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5) $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$$\overrightarrow{AC} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = 2\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} -1 \\ 3 \\ -1 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\prod : \underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 5$$

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6)

Method 1

Take a point in $\prod_1$, say C (13,0,0) which satisfies $r \cdot \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 13$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 12 \\ -7 \\ 10 \end{pmatrix}$$

Proj $\overrightarrow{AC}$ on $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = \left| \dfrac{\overrightarrow{AC} \cdot \mathbf{n}}{|\mathbf{n}|} \right| = \left| \begin{pmatrix} 12 \\ -7 \\ 10 \end{pmatrix} \cdot \dfrac{1}{\sqrt{21}} \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \right|$

$= 2\sqrt{21}$

Method 2

This method useful for finding the foot of perpendicular

Vector equation of a line through A and parallel to $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$: $\mathbf{r} = \begin{pmatrix} 1 \\ 7 \\ -10 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \qquad \lambda \in R$

Let C be the foot of the perpendicular from A to $\prod_1$

Then $\overrightarrow{OC} = \begin{pmatrix} 1+\lambda \\ 7+2\lambda \\ -10-4\lambda \end{pmatrix}$

C lies on $\prod_1$: $\begin{pmatrix} 1+\lambda \\ 7+2\lambda \\ -10-4\lambda \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} = 13$

$\Rightarrow \lambda = -2$

$\overrightarrow{OC} = \begin{pmatrix} 1+\lambda \\ 7+2\lambda \\ -10-4\lambda \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$; $\overrightarrow{AC} = \begin{pmatrix} 1+\lambda-1 \\ 7+2\lambda-7 \\ -10-4\lambda+10 \end{pmatrix}$

$\Rightarrow$ d $=$ $AC = \sqrt{84} = 2\sqrt{21}$

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7) A line through B perpendicular to plane $\prod$ is

$$\underset{\sim}{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}, \quad \alpha \in \mathbb{R} \quad . \ (1)$$

Sub (1) into equation of plane:

$$\begin{pmatrix} 2+\alpha \\ 3-\alpha \\ -1-2\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 9$$

$$2+\alpha-3+\alpha+2+4\alpha = 9$$

$$\therefore \ \alpha = \frac{4}{3}$$

Hence position vector $\overrightarrow{OD}$ of foot of perpendicular from B to $\prod$ is

$$\overrightarrow{OD} = \begin{pmatrix} 2+\frac{4}{3} \\ 3-\frac{4}{3} \\ -1-\frac{8}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 10 \\ 5 \\ -11 \end{pmatrix}$$

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8) At the point of intersection P, $\begin{pmatrix} 1+l \\ 1 \\ 3-2l \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 4$

$\Rightarrow 1 + l + 2 - 3 + 2\,l = 4 \Rightarrow l = \frac{4}{3}$ Therefore $\begin{pmatrix} 1+\frac{4}{3} \\ 1 \\ 3-\frac{8}{3} \end{pmatrix} = \begin{pmatrix} \frac{7}{3} \\ 1 \\ \frac{1}{3} \end{pmatrix}$

The point of intersection is $\left(\frac{7}{3}, 1, \frac{1}{3}\right)$.

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9) **Method 1**

Line passing through Q and perpendicular to Π_1,

$$l : \mathbf{r} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \alpha \in \mathbb{R}$$

Since R lies on l, $\overrightarrow{OR} = \begin{pmatrix} 4+2\alpha \\ -2\alpha \\ 8+\alpha \end{pmatrix}$ for some $\alpha \in \mathbb{R}$

At point of intersection of l and Π_1, R

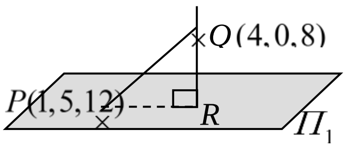
$$\begin{pmatrix} 4+2\alpha \\ -2\alpha \\ 8+\alpha \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = 4$$

$$\Rightarrow 8 + 4\alpha + 4\alpha + 8 + \alpha = 4$$

$$\Rightarrow \alpha = -\tfrac{4}{3}$$

$$\overrightarrow{OR} = \begin{pmatrix} 4/3 \\ 8/3 \\ 20/3 \end{pmatrix} \text{ (shown)}$$

Method 2



$\overrightarrow{RQ}$ = Projection of $\overrightarrow{PQ}$ onto the normal

$$= \left(\frac{\overrightarrow{PQ} \cdot \mathbf{n}_1}{|\mathbf{n}_1|} \right) \frac{\mathbf{n}_1}{|\mathbf{n}_1|}$$

$$= \left(\frac{\begin{pmatrix} 3 \\ -5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{4+4+1}} \right) \frac{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}{\sqrt{4+4+1}}$$

$$= \left(\frac{4}{3} \right) \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8/3 \\ -8/3 \\ 4/3 \end{pmatrix}$$

$$\overrightarrow{OR} = \overrightarrow{OQ} - \overrightarrow{RQ} = \begin{pmatrix} 4 \\ 0 \\ 8 \end{pmatrix} - \begin{pmatrix} 8/3 \\ -8/3 \\ 4/3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 8/3 \\ 20/3 \end{pmatrix} \text{ (shown)}$$

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10) **<u>Method 1:</u>**

$-2x - y + 5z = 3$

$2x + y + 2z = 4$

Using GC, $2x + y = 2$; $z = 1$.

When $x = \lambda$, $y = 2 - 2\lambda$ $\Rightarrow \underset{\sim}{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}.$

<u>Method 2:</u>

Normal of l : $\begin{pmatrix} -2 \\ -1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -7 \\ 14 \\ 0 \end{pmatrix} = -7 \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$

Let $x = 0$, $-y + 5z = 3$; $y + 2z = 4$.

Solving simultaneously, $y = 2$, $z = 1$.

$$\Rightarrow \underset{\sim}{r} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \lambda \in \mathbb{R}.$$

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11)

Π_1 : $x + 2y - 4z = 13$ ……. (1)

Π_2 : $x + 3y + 3z = -8$ ……. (2)

By G.C. solve equations (1) & (2)

The vector equation of the line of intersection is

l : $\mathbf{r} = \begin{pmatrix} 55 \\ -21 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 18 \\ -7 \\ 1 \end{pmatrix}$ where $\lambda \in \mathbb{R}$. or $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 18 \\ -7 \\ 1 \end{pmatrix}$ etc

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12) $\Pi_1 : \underset{\sim}{r} \bullet \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} = 14$

$\Pi_2 : \underset{\sim}{r} \bullet \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 5$

Direction vector of L = $\begin{pmatrix}1\\1\\2\end{pmatrix}\times\begin{pmatrix}5\\-2\\1\end{pmatrix}=\begin{pmatrix}5\\9\\-7\end{pmatrix}$

$$x+y+2z=5\cdots\cdots\cdots\cdots(1)$$
$$5x-2y+z=14\cdots\cdots\cdots\cdots(2)$$

Put $x = 2$,

from (1), $y+2z=3\cdots\cdots\cdots(3)$

from (2), $-2y+z=4\cdots\cdots\cdots(4)$

$(3)-2(4)\Rightarrow 5y=-5$

$\Rightarrow y=-1$

$\Rightarrow z=4+2(-1)=2$

$$\therefore L:\underset{\sim}{r}=\begin{pmatrix}2\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}5\\9\\-7\end{pmatrix},\ \lambda\in\Re$$

$$\therefore\quad p=-1,\ q=2,\ r=5,\ s=9$$

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13) Angle between π_1 and π_2

$$=\cos^{-1}\left|\frac{\begin{pmatrix}2\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}4\\3\\-1\end{pmatrix}}{\sqrt{2^2+1^2+2^2}\sqrt{26}}\right|$$

$$=\cos^{-1}\left|\frac{8-3-2}{3\sqrt{26}}\right|$$

$= 78.7^{o}$

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14)

$$\cos\theta=\frac{\left|\begin{pmatrix}2\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\2\end{pmatrix}\right|}{\sqrt{9}\sqrt{9}}$$

$$\cos\theta=\frac{4}{9}$$

$\theta=64^{\circ}$ (nearest degree)

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15) Angle between $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and $\begin{pmatrix}1\\0\\-2\end{pmatrix}$

$$=\cos^{-1}\left(\frac{\begin{pmatrix}1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix}}{\sqrt{1^2+2^2+(-1)^2}\sqrt{1^2+0^2+(-2)^2}}\right)$$

$$=\cos^{-1}\frac{3}{\sqrt{30}}=56.8^{\circ}$$

$\therefore$ Angle between l and Π is $90^{\circ}-56.8^{\circ}$

$= 33.2^{\circ}$ (1d.p.)

16) Let $\lambda = 2$,

$\overrightarrow{OA} = \mathbf{j} + \mathbf{k} + 2(\mathbf{i} - 2\mathbf{j} + \mathbf{k}) = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$.

$\overrightarrow{AB} = (2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) - (2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$= 4\mathbf{j} + \mathbf{k}$

$$x + 2y + 3z = 5 \Rightarrow \mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5$$

Angle between $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$

$$= \cos^{-1}\left(\frac{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}}{\sqrt{1^2 + 2^2 + 3^2}\sqrt{4^2 + 1^2}} \right)$$

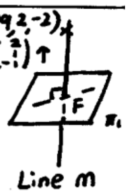
$$= \cos^{-1} \frac{11}{\sqrt{14}\sqrt{17}} = 44.5°$$

Acute angle between the line *AB* and the plane π_1 = 90 - 44.5° = 45.5°

17)

(13) i) π_1 : $\underline{r} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 4$

Line m : $\underline{r} = \begin{pmatrix} 9 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\lambda \in \mathbb{R}$



At F, $\left[\begin{pmatrix} 9 \\ 2 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 4$

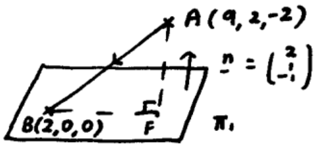
$22 + 6\lambda = 4$

$\lambda = -3$

$\overrightarrow{OF} = \begin{pmatrix} 9 \\ 2 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

∴ Foot of perpendicular, F (3, -1, 1).

<u>OR</u>



Choose a pt B in π_1 such that $\overrightarrow{OB} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$.

$\overrightarrow{AF}$ = projection of $\overrightarrow{AB}$ on $\underline{n}$

$= (\overrightarrow{AB} \cdot \hat{\underline{n}})\hat{\underline{n}}$

$= \left[\begin{pmatrix} -7 \\ -2 \\ 2 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right] \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$= -\frac{18}{6} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$= -3 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\overrightarrow{OF} = \begin{pmatrix} 9 \\ 2 \\ -2 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

∴ Foot of perpendicular, F(3, -1, 1).

⑬ ii) $\begin{pmatrix}-3\\2\\-7\end{pmatrix} \times \begin{pmatrix}1\\-1\\4\end{pmatrix} = \begin{pmatrix}1\\5\\1\end{pmatrix}$

$\pi_2 : \begin{pmatrix}2\\1\\-2\end{pmatrix} \cdot \begin{pmatrix}1\\5\\1\end{pmatrix} = 2+5-2 = 5$

$\therefore \underset{\sim}{r} \cdot \begin{pmatrix}1\\5\\1\end{pmatrix} = 5$

$2x + y - z = 4$

$x + 5y + z = 5$

From GC : $x = \frac{5}{3} + \frac{2}{3}z$

$y = \frac{2}{3} - \frac{1}{3}z$

$z = z$

$\therefore \ l : \underset{\sim}{r} = \begin{pmatrix}5/3\\2/3\\0\end{pmatrix} + \lambda \begin{pmatrix}2\\-1\\3\end{pmatrix}, \lambda \in R$

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18) $\Pi_1 : \mathbf{r}. \begin{pmatrix}5\\-3\\-4\end{pmatrix} = 2$

$\Pi_2 : \mathbf{r}. \begin{pmatrix}2\\1\\5\end{pmatrix} = 15$

(i) $\begin{pmatrix}5\\-3\\-4\end{pmatrix} \bullet \begin{pmatrix}2\\1\\5\end{pmatrix} = 10 - 3 - 20 = -13$

$$\cos\theta = \frac{\begin{pmatrix}5\\-3\\-4\end{pmatrix} \bullet \begin{pmatrix}2\\1\\5\end{pmatrix}}{\left|\begin{pmatrix}5\\-3\\-4\end{pmatrix}\right|\left|\begin{pmatrix}2\\1\\5\end{pmatrix}\right|} = \frac{-13}{\sqrt{50}\sqrt{30}}$$

$\theta = 109.6°$

The acute angle between the planes = 70° (nearest degree)

(ii) $\begin{pmatrix}5\\-3\\-4\end{pmatrix} \times \begin{pmatrix}2\\1\\5\end{pmatrix} = \begin{pmatrix}-11\\-33\\11\end{pmatrix}$ parellel to $\begin{pmatrix}-1\\-3\\1\end{pmatrix}$

Equation of the line through A and parallel to the line of intersection is

$$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + \alpha \begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}, \text{ where } \alpha \in \mathbb{R}$$

(iii) Let the equation of the line through A perpendicular to Π_2 be

$$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$$

At the point of intersection,

$$\left(\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \right) . \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = 15$$

$$2(1+2\lambda) + (-2 + \lambda) +5 (-3+5\lambda) = 15$$

$$\lambda = 1$$

The position vector of the foot of the perpendicular from A to Π_2 is $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.