Vectors Revision Worksheet (Lines)

1. $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Angle between the two vectors

$$= \cos^{-1}\left(\frac{\begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix} . \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}{\sqrt{36}\sqrt{3}} \right)$$

$$= \cos^{-1}\left(\frac{-1}{\sqrt{3}} \right) = 125.3°$$

Vector equation of the line *l* passing through A and B is

$$\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -5 \end{pmatrix}, \lambda \in \square$$

In Cartesian form,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 5 \\ -5 \end{pmatrix}$$

$$\therefore x - 1 = \frac{y+1}{5} = \frac{1-z}{5}$$

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2. $\overrightarrow{OP} = \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$, $\overrightarrow{OQ} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$

Angle OPQ

$$= \cos^{-1}\left(\frac{\overrightarrow{PO}.\overrightarrow{PQ}}{\left|\overrightarrow{PO}\right|\left|\overrightarrow{PQ}\right|} \right)$$

$$= \cos^{-1}\left(\frac{\begin{pmatrix}-3\\2\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\5\\2\end{pmatrix}}{\sqrt{14}\sqrt{30}}\right)$$

$$= \cos^{-1}\left(\frac{15}{\sqrt{420}}\right) = 43.0° \text{ (1 dp)}$$

Vector equation of the line *l* passing through P and Q is

$$\mathbf{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\5\\2\end{pmatrix}, \lambda \in \square$$

In Cartesian form,

$$\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}-1\\5\\2\end{pmatrix}$$

$$\therefore 2 - x = \frac{y-3}{5} = \frac{z-1}{2}$$

3

L_1: $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
L_2: $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + \mathbf{j})$.

If L_1 and L_2 intersect,
$3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + \mathbf{j})$

$$3 + s = 2 + 2t$$
$$1 + 2s = 2 + t$$
$$2 - s = 1 \Rightarrow s = 1$$

When $s = 1$, $t = 1$

Therefore $2\mathbf{i} + 2\mathbf{j} + \mathbf{k} + 1(2\mathbf{i} + \mathbf{j}) = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$
Hence point of intersection is (4, 3, 1).

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4. L_1: $\mathbf{r} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} - \mathbf{k})$
   L_2: $\mathbf{r} = \mathbf{2i} + 2\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

Since $(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \neq m\,(2\mathbf{i} + \mathbf{j} + \mathbf{k})$ for some $m \in \square$, L_1 is not parallel to L_2.

If L_1 and L_2 intersect,
$3\mathbf{i} + \mathbf{j} + 2\mathbf{k} + s(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) = \mathbf{2i} + 2\mathbf{j} + \mathbf{k} + t(2\mathbf{i} + \mathbf{j} + \mathbf{k})$

$3+s=2+2t\ldots(1)$

$1+2s=2+t\ldots(2)$

$2-s=1+t\ldots(3)$

Solving (1) and (3),
$5 = 3 + 3t \Rightarrow t = 2/3$
$s = 2 + 4/3 - 3 = 1/3$

Sub into (2)
LHS = 1 + 2(1/3) = 5/3
RHS = 2 + 2/3 = 8/3

Since LHS ≠ RHS, L_1 and L_2 do not intersect.
Hence L_1 and L_2 are skew lines.

5.
If lines intersect,
5**i** + 2**j** + 4**k** +λ(**i** + 3**j** + **k**)=3**i** + **j** + **k** + μ(4**i** + 7**j** + 5**k**)

$5+\lambda=3+4\mu\ldots(1)$

$2+3\lambda=1+7\mu\ldots(2)$

$4+\lambda=1+5\mu\ldots(3)$

Solving (1) and (3), $1 = 2 - \mu \Rightarrow \mu = 1$
When $\mu = 1$, $\lambda = 2$

Therefore position vector point of intersection = 3**i** + **j** + **k** + 1(4**i** + 7**j** + 5**k**)
= 7**i** + 8**j** + 6**k**

Acute angle between the lines = $\cos^{-1}\left|\dfrac{\begin{pmatrix}1\\3\\1\end{pmatrix}.\begin{pmatrix}4\\7\\5\end{pmatrix}}{\sqrt{11}\sqrt{90}}\right|$ = 17.5° (1 dp)

6.
Perpendicular distance from O to the line

$$=\left|\frac{\begin{pmatrix}4\\2\\5\end{pmatrix}\times\begin{pmatrix}2\\-2\\1\end{pmatrix}}{\sqrt{9}}\right|=\left|\frac{\begin{pmatrix}12\\6\\-12\end{pmatrix}}{3}\right|=6$$

7.

Perpendicular distance from A to the line

$$= \left| \frac{\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}}{\sqrt{6}} \right| = \left| \frac{\begin{pmatrix} 2 \\ -4 \\ -3 \end{pmatrix}}{\sqrt{6}} \right| = \frac{\sqrt{29}}{\sqrt{6}}$$

8.

Vector equation of the line passing through A and B is

$$\mathbf{r} = \begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$$

Since P is on line AB,

$$\overrightarrow{OP} = \begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}, \text{ for some } \lambda \in \mathbb{R}$$

Since OP is perpendicular to AB,

$$\overrightarrow{OP} \cdot \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} = 0$$

$$\left[\begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix} = 0$$

$$4 + \lambda - 35 + 25\lambda + 4 + \lambda = 0$$

$$\lambda = 1$$

$$\therefore \overrightarrow{OP} = \begin{pmatrix} 4 \\ 7 \\ 4 \end{pmatrix} + (1) \begin{pmatrix} 1 \\ -5 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ 2 \\ 5 \end{pmatrix}$$

9.

Since N is on line,

$$\overrightarrow{ON} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}, \text{ for some } \lambda \in \mathbb{R}$$

Since PN is perpendicular to AB,

$$\overrightarrow{PN} \cdot \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} = 0$$

$$\left[\begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} = 0$$

$$\left[\begin{pmatrix} 0 \\ -2 \\ 13 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix} = 0$$

$$4\lambda - 10 + 25\lambda + 52 + 16\lambda = 0$$

$$45\lambda = -42$$

$$\lambda = -\frac{14}{15}$$

$$\overrightarrow{ON} = \begin{pmatrix} 2 \\ 1 \\ 9 \end{pmatrix} - \frac{14}{15} \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2/15 \\ -11/3 \\ 79/15 \end{pmatrix}$$

10.(i)

$$x - 1 = \frac{y+1}{2} = \frac{z+1}{-2} = \lambda$$

$$x = 1 + \lambda \text{ , } y = -1 + 2\lambda \text{ , } z = -1 - 2\lambda$$

An vector equation of l_1 is $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \lambda \in \mathbb{R}$

Vector equation of the line passing through A and B is

$$\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ -8 \end{pmatrix}, \mu \in \mathbb{R}$$

(ii)

Since ($\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$) $\neq$ m ($\mathbf{i} - \mathbf{j} - 8\mathbf{k}$) for some m $\in \mathbb{R}$, lines are not parallel.

If l_1 and l_2 intersect,

$$\begin{pmatrix}1\\-1\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+\mu\begin{pmatrix}1\\-1\\-8\end{pmatrix}$$

$1+\lambda=1+\mu ...(1)$

$-1+2\lambda=-2-\mu ..(2)$

$-1-2\lambda=3-8\mu ...(3)$

Solving (1) and (2),
$3\lambda = -1 \Rightarrow \lambda = -1/3$
$\mu = -1/3$

Sub into (3)
LHS = -1 - 2(-1/3) = -1/3
RHS = 3 -8(-1/3) = 17/3

Since LHS $\neq$ RHS, l_1 and l_2 do not intersect.
Hence l_1 and l_2 are skew lines.

(iii) Length of projection of $\overrightarrow{AB}$ on l_1

$$=\frac{\left|\begin{pmatrix}1\\-1\\-8\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}\right|}{\sqrt{9}}=\frac{15}{3}=5$$

(iv)
Since N is on line,

$$\overrightarrow{ON} = \begin{pmatrix}1\\-1\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}, \text{ for some } \lambda\in\mathbb{R}$$

Since AN is perpendicular to line,

$$\overrightarrow{AN}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=0$$

$$\left[\begin{pmatrix}1\\-1\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}-\begin{pmatrix}1\\-2\\3\end{pmatrix}\right]\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=0$$

$$\left[\begin{pmatrix}0\\1\\-4\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}\right]\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=0$$

$\lambda+2+4\lambda+8+4\lambda=0$
$9\lambda=-10$

$\lambda = -\frac{10}{9}$

$\overrightarrow{ON} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} - \frac{10}{9}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} -1/9 \\ -29/9 \\ 11/9 \end{pmatrix}$

$\overrightarrow{AN} = \begin{pmatrix} -1/9 \\ -29/9 \\ 11/9 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} -10/9 \\ -11/9 \\ -16/9 \end{pmatrix}$

Shortest distance between the point A and l_1

$= \frac{1}{9}\sqrt{100+121+256} = \frac{\sqrt{477}}{9}$

11(i)
Vector equation of the line l passing through A and B is

$\mathbf{r} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}, \lambda \in \mathbb{R}$

In Cartesian form,

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 10 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$

$\therefore \frac{x-5}{3} = \frac{z-10}{4}, y = 4$

(ii) Length of projection of $\overrightarrow{AC}$ on line AB

$= \frac{\left| \begin{pmatrix} -10 \\ 5 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \right|}{\sqrt{25}} = \frac{|-50|}{5} = 10$

(iii) $|\overrightarrow{AC}| = \sqrt{100+25+25} = \sqrt{150} = 5\sqrt{6}$

Perpendicular distance from C to the line AB

$= \sqrt{150-100} = \sqrt{50} = 5\sqrt{2}$

$$\overrightarrow{AN} = -10\frac{\begin{pmatrix}3\\0\\4\end{pmatrix}}{\sqrt{25}} = \begin{pmatrix}-6\\0\\-8\end{pmatrix}$$

$$\therefore \overrightarrow{ON} = \overrightarrow{AN} + \overrightarrow{OA} = \begin{pmatrix}-6\\0\\-8\end{pmatrix} + \begin{pmatrix}5\\4\\10\end{pmatrix} = \begin{pmatrix}-1\\4\\2\end{pmatrix}$$