Vectors Revision

2

$$\vec{OP} = \frac{1}{3}\left[\vec{OB} + 2\vec{OC}\right] = \frac{1}{3}\left[\begin{pmatrix}2\\1\\0\end{pmatrix} + 2\begin{pmatrix}2\\1\\3\end{pmatrix}\right]$$ **[M1]**

$$= \begin{pmatrix}2\\1\\2\end{pmatrix}$$ **[A1]**

$$\vec{PA} = \begin{pmatrix}3\\1\\1\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$ **[M1]**

$$\vec{PB} = \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\-2\end{pmatrix}$$ **[M1]**

Area of triangle APB $= \frac{1}{2}\left|\vec{PA} \times \vec{PB}\right| = \frac{1}{2}\left|\begin{pmatrix}1\\0\\-1\end{pmatrix} \times \begin{pmatrix}0\\0\\-2\end{pmatrix}\right|$ **[M1]**

$$= \frac{1}{2}\left|\begin{pmatrix}0\\2\\0\end{pmatrix}\right|$$ **[M1]**

$$= 1$$ **[A1]**

3 (i) $l: \mathrm{r} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

$$\overrightarrow{BC} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 0$$

$$\overrightarrow{BC} = \begin{pmatrix} 2+3\lambda \\ 2+\lambda \\ 2+4\lambda \end{pmatrix} - \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2+3\lambda \\ \lambda \\ 5+4\lambda \end{pmatrix}$$

$$\begin{pmatrix} 2+3\lambda \\ \lambda \\ 5+4\lambda \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 0 \Rightarrow 6+9\lambda+\lambda+20+16\lambda = 0$$

$$26\lambda = -26$$

$$\lambda = -1$$

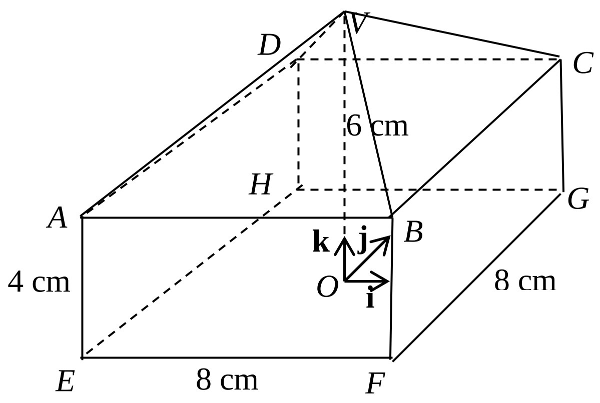
$$\overrightarrow{OC} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

(ii) perpendicular distance $= \dfrac{\left|\overrightarrow{AC} \times \overrightarrow{AB}\right|}{\left|\overrightarrow{AB}\right|}$

$$= \frac{\left|\begin{pmatrix} -3 \\ -1 \\ -4 \end{pmatrix} \times \begin{pmatrix} -2 \\ 0 \\ -5 \end{pmatrix}\right|}{\left|\begin{pmatrix} -2 \\ 0 \\ -5 \end{pmatrix}\right|} = \frac{\left|\begin{pmatrix} 5 \\ -7 \\ -2 \end{pmatrix}\right|}{\sqrt{4+25}}$$

$$= \frac{\sqrt{78}}{\sqrt{29}}$$

4



(i) Find the acute angle between the vectors $\overrightarrow{BV}$ and $\overrightarrow{CE}$. **[4]**

(ii) Find a vector equation of the line *CE*. **[1]**

(iii) Find the position vector of the foot of the perpendicular from *V* to the line *CE*, leaving your answer in the exact form. **[4]**

(i) $\overrightarrow{OB} = \begin{pmatrix} 4 \\ -4 \\ 4 \end{pmatrix}$, $\overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix}$, $\overrightarrow{OE} = \begin{pmatrix} -4 \\ -4 \\ 0 \end{pmatrix}$

$$\overrightarrow{BV} = \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \quad \overrightarrow{CE} = -4\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

$$\cos\theta = \frac{2\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \bullet (-4)\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}{(2)(4)\sqrt{9}\sqrt{9}} = -\frac{1}{9} \quad \Rightarrow \quad \theta = 96.4^\circ \text{, required angle is } 83.6^\circ$$

(ii) $\mathbf{r} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ or $\mathbf{r} = \begin{pmatrix} -4 \\ -4 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$

(iii) Let the foot of the perpendicular be *N*.

Using $\overrightarrow{ON} = \begin{pmatrix} 4 \\ 4 \\ 4 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $\overrightarrow{VN} = \begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 4+2\lambda \\ 4+2\lambda \\ -2+\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\lambda = -\frac{14}{9} \quad , \qquad \overrightarrow{ON} = \begin{pmatrix} \frac{8}{9} \\ \frac{8}{9} \\ \frac{22}{9} \end{pmatrix}$$

OR : Using $\overrightarrow{ON} = \begin{pmatrix} -4 \\ -4 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, $\overrightarrow{VN} = \begin{pmatrix} -4 \\ -4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} -4+2\lambda \\ -4+2\lambda \\ -6+\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 0$$

$$\lambda = \frac{22}{9} \quad , \qquad \overrightarrow{ON} = \begin{pmatrix} \frac{8}{9} \\ \frac{8}{9} \\ \frac{22}{9} \end{pmatrix}$$

5 (i) $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ **[M1]**

$\therefore$ Vector equation of line l_2 : $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, $\lambda \in \Re$ **[M1]**

l_1 : $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -5 \end{pmatrix}$

At point of intersection,

$\begin{pmatrix} 1+\lambda \\ 2+\lambda \\ 3-\lambda \end{pmatrix} = \begin{pmatrix} 1+2\mu \\ 1+\mu \\ -5\mu \end{pmatrix}$ for some $\lambda, \mu \in \Re$ **[M1]**

$$\Rightarrow \begin{cases} 1+\lambda = 1+2\mu & ---(1) \\ 2+\lambda = 1+\mu & ---(2) \\ 3-\lambda = -5\mu & ---(3) \end{cases}$$

From (2),

$\mu = 1+\lambda$

Subst. into (1),

$\mu = 1+2\mu$

$\Rightarrow \mu = -1$ **[B1]**

$\Rightarrow \lambda = -2$

When μ = -1, λ = -2,

LHS of (3) = 3 + 2 = 5

RHS of (3) = -5(-1)

= 5

= LHS of (3) **[M1]**

$\therefore$ The lines l_1 and l_2 intersect. (Shown)

$\overrightarrow{OM} = \begin{pmatrix} 1-2 \\ 2-2 \\ 3+2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$ **[M1]**

$\therefore$ Coordinates of M = (-1, 0, 5) **[A1]**

(ii) Let the angle be θ.

$$\cos\theta = \frac{\begin{pmatrix}1\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-5\end{pmatrix}}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+1^2+5^2}}$$ **[M2]**

$$= \frac{8}{\sqrt{3}\sqrt{30}}$$

$\Rightarrow \theta = 32.5^\circ$ (corr. to nearest 0.1^o) **[A1]**

OR

$$\cos\theta = \frac{\begin{pmatrix}-1\\-1\\1\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-5\end{pmatrix}}{\sqrt{1^2+1^2+1^2}\sqrt{2^2+1^2+5^2}}$$ **[M2]**

$$= \frac{-8}{\sqrt{3}\sqrt{30}}$$

$\Rightarrow \theta = 147.5^\circ$ (corr. to nearest 0.1^o) **[A1]**

(iii) N lies on $l_1 \Rightarrow \vec{ON} = \begin{pmatrix}1+2\mu\\1+\mu\\-5\mu\end{pmatrix}$ for some $\mu \in \Re$ **[M1]**

$$\vec{AN} = \begin{pmatrix}1+2\mu\\1+\mu\\-5\mu\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix}$$ **[M1]**

$$= \begin{pmatrix}2\mu\\-1+\mu\\-3-5\mu\end{pmatrix}$$

$$AN \perp l_1 \Rightarrow \begin{pmatrix}2\mu\\-1+\mu\\-3-5\mu\end{pmatrix}\bullet\begin{pmatrix}2\\1\\-5\end{pmatrix} = 0$$ **[M1]**

$$\Rightarrow 4\mu - 1 + \mu + 15 + 25\mu = 0$$

$$\Rightarrow \mu = -\frac{7}{15}$$ **[M1]**

$$\therefore \vec{ON} = \begin{pmatrix}1-\frac{14}{15}\\1-\frac{7}{15}\\-5\left(-\frac{7}{15}\right)\end{pmatrix} = \begin{pmatrix}\frac{1}{15}\\\frac{8}{15}\\\frac{7}{3}\end{pmatrix}$$ **[A1]**

(iv) Length of projection of $\vec{MA}$ on line $l_1 = MN$

$$\vec{MN} = \begin{pmatrix} \frac{1}{15} \\ \frac{8}{15} \\ \frac{7}{3} \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 5 \end{pmatrix}$$ **[M1]**

$$= \begin{pmatrix} \frac{16}{15} \\ \frac{8}{15} \\ -\frac{8}{3} \end{pmatrix}$$

$$\therefore MN = \sqrt{\left(\frac{16}{15}\right)^2 + \left(\frac{8}{15}\right)^2 + \left(\frac{8}{3}\right)^2}$$ **[M1]**

$= 2.92$ **[A1]**

6 **(i)**

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}$$

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix}$$

$$\mathbf{r} \cdot \begin{pmatrix} 3 \\ -3 \\ -1 \end{pmatrix} = -2$$

$\therefore\ \Pi_2 : 3x - 3y - z = -2$

$\Pi_1 : 2x + y + 3z = 5$

$$m : \mathbf{r} = \begin{pmatrix} \frac{13}{9} \\ \frac{19}{9} \\ 0 \end{pmatrix} + \gamma' \begin{pmatrix} -\frac{8}{9} \\ -\frac{11}{9} \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{13}{9} \\ \frac{19}{9} \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 8 \\ 11 \\ -9 \end{pmatrix}$$

Alternatively

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$\left[\begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right] \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 5$$

$$23 + 3\lambda + 14\mu = 5$$

$$\lambda = -6 - \frac{14}{3}\mu$$

$$\mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix} + \left(-6 - \frac{14}{3}\mu \right) \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} -\frac{8}{3} \\ -\frac{11}{3} \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -3 \\ -4 \\ 5 \end{pmatrix} + \gamma \begin{pmatrix} 8 \\ 11 \\ -9 \end{pmatrix}$$

(ii) $m \mathbin{//} \Pi_3$. Hence $\begin{pmatrix} 8 \\ 11 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} a \\ 2 \\ -2 \end{pmatrix} = 0$

$$a = -5$$

$$\Pi_3 : \ \mathbf{r} \cdot \begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix} = b$$

$$\mathbf{r} \cdot \frac{\begin{pmatrix} -5 \\ 2 \\ -2 \end{pmatrix}}{\sqrt{33}} = \frac{b}{\sqrt{33}}$$

$$\left| \frac{b}{\sqrt{33}} \right| = 4$$

$$b = 4\sqrt{33} \quad \text{or} \quad -4\sqrt{33}$$

(iii) $\sin\theta = \sqrt{\frac{5}{7}} \ \Rightarrow \ \cos\theta = \sqrt{\frac{2}{7}}$

$$\frac{\left| \begin{pmatrix} a \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \right|}{\sqrt{a^2 + 8}\sqrt{14}} = \sqrt{\frac{2}{7}}$$

$$|2a-4| = 2\sqrt{a^2+8}$$
$$a = -1$$

7

(i) $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ 7 \\ -2 \end{pmatrix}$

length of projection of

$$\overrightarrow{AB} \text{ onto } l_1 = \left| \begin{pmatrix} -7 \\ 7 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right| \frac{1}{\sqrt{3}} = \left| \frac{-2}{\sqrt{3}} \right| = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.15$$

(ii)

$$\cos\theta = \frac{\begin{pmatrix} -7 \\ 7 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{102}\sqrt{2}} = \frac{-5}{\sqrt{204}}$$

$\theta = 110.49°$

acute angle = $110.49^\circ - 90.0° = 20.5°$ (1 dp)

(iii) $\overrightarrow{OQ} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ for some value of t

$$\begin{pmatrix} -3+t \\ 10 \\ 3-t \end{pmatrix} \square \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = -1$$

$$-3+t-3+t = -1$$

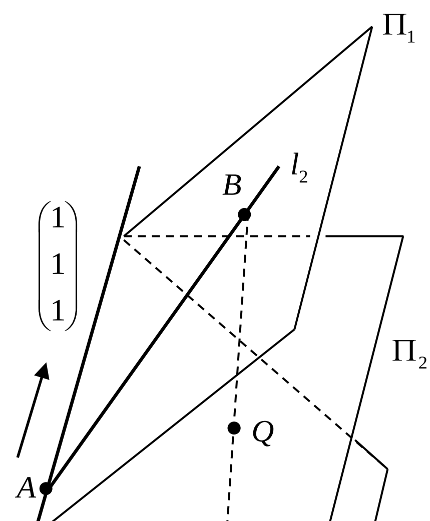
$$t = \frac{5}{2}$$

$$\overrightarrow{OQ} = \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} + \frac{5}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 20 \\ 1 \end{pmatrix}$$

(iv) Let the point of reflection of B in Π_2 to be P

$$\overrightarrow{OQ} = \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OP}\right)$$

$$\overrightarrow{OP} = 2\overrightarrow{OQ} - \overrightarrow{OB} = \begin{pmatrix} -1 \\ 20 \\ 1 \end{pmatrix} - \begin{pmatrix} -3 \\ 10 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix}$$



$$\overrightarrow{AP} = \begin{pmatrix} 2 \\ 10 \\ -2 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -7 \end{pmatrix}$$

$$\overrightarrow{AP} \times \mathbf{d}_1 = \begin{pmatrix} -2 \\ 7 \\ -7 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix}$$

$$\mathbf{r} \cdot \begin{pmatrix} -14 \\ 5 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -14 \\ 5 \\ 9 \end{pmatrix} = 4 \text{ or } \mathbf{r} \cdot \begin{pmatrix} 14 \\ -5 \\ -9 \end{pmatrix} = -4 \text{ (can also use point } P\text{)}$$