Vectors – Scalar and Vector Products

1. $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$\mathbf{a \cdot b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 6+2+3 = 11$

Length of projection of **a** on **b**

$= \frac{\mathbf{a.b}}{|\mathbf{b}|} = \frac{11}{\sqrt{9+4+1}} = \frac{11}{\sqrt{14}}$

---

2. $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$,

$\overrightarrow{BC} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Length of projection of AB on BC

$= \frac{\overrightarrow{AB} \cdot \overrightarrow{BC}}{|\overrightarrow{BC}|}$

$= \frac{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}{\sqrt{1+1+4}} = \frac{3}{\sqrt{6}} = \frac{\sqrt{6}}{2}$

---

3. $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 2 - 3 + 1 = 0$

Hence **a** and **b** are perpendicular.

---

4. $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = -3 + 2 + 2 = 1 \neq 0$

Hence **a** and **b** are not perpendicular to each other.

5. $\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(1)-(-1)(1) \\ -[2(1)-(1)(1)] \\ 2(-1)-1(3) \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$

---

6. $\begin{pmatrix} 2 \\ 5 \\ -2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 16 \\ -14 \\ -19 \end{pmatrix}$

---

7. $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}$

$\overrightarrow{PR} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Area of triangle PQR

$= ½ \, |\overrightarrow{PQ} \times \overrightarrow{PR}|$

$= \frac{1}{2}\left|\begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}\right|$

$= \frac{1}{2}\left|\begin{pmatrix} -6 \\ 6 \\ -3 \end{pmatrix}\right| = \frac{\sqrt{81}}{2} = \frac{9}{2}$ units^2

---

8. $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$\overrightarrow{AC} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$

Area of triangle ABC

$= ½ \, |\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \frac{1}{2}\left|\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}\right|$

$= \frac{1}{2}\left|\begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}\right| = \frac{\sqrt{21}}{2}$ units^2

9. $\mathbf{p}\times\mathbf{q} = 3\mathbf{p}\times\mathbf{r}$

$\mathbf{p}\times(\mathbf{q}-3\mathbf{r}) = \mathbf{0}$

$\mathbf{p}\,//\,(\mathbf{q}-3\mathbf{r})$

$\mathbf{q}-3\mathbf{r} = \lambda\mathbf{p}$, where λ is a scalar.

$$(\mathbf{q}-3\mathbf{r}).(\mathbf{q}-3\mathbf{r}) = \lambda\mathbf{p}.\lambda\mathbf{p}$$

$$|\mathbf{q}|^2 - 6\mathbf{q}.\mathbf{r} + 9|\mathbf{r}|^2 = \lambda^2|\mathbf{p}|^2$$

$$5^2 - 6\times5\times2\left(\frac{5}{6}\right) + 9(2)^2 = \lambda^2(1)^2$$

$$\lambda^2 = 11$$

$$\lambda = \pm\sqrt{11}$$

10.

(a) $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{a} \Rightarrow \mathbf{a}\times\mathbf{b} - \mathbf{b}\times\mathbf{a} = \mathbf{0} \Rightarrow 2\mathbf{a}\times\mathbf{b} = \mathbf{0}\ \therefore \mathbf{a}\times\mathbf{b}$ is the zero vector.

(b) Since the vector perpendicular to both **a** ($\overrightarrow{OA}$) and **b** ($\overrightarrow{OB}$) is also perpendicular to **c** $\overrightarrow{OC}$, $(\mathbf{a}\times\mathbf{b}).\mathbf{c} = \mathbf{0}$ implies that the four points are coplanar.

11(i)

$$\overrightarrow{OM} = \frac{2\overrightarrow{OB}+\overrightarrow{OA}}{3}$$

$$= \frac{2\begin{pmatrix}3\\3\\-3\end{pmatrix}+\begin{pmatrix}-3\\0\\9\end{pmatrix}}{3}$$

$$= \begin{pmatrix}1\\2\\1\end{pmatrix}$$

(ii) If *OM* is perpendicular to *AB*, $\overrightarrow{OM}\cdot\overrightarrow{AB} = 0$

$$\overrightarrow{AB} = \begin{pmatrix}6\\3\\-12\end{pmatrix}$$

$$\overrightarrow{OM}\cdot\overrightarrow{AB} = \begin{pmatrix}1\\2\\1\end{pmatrix}\cdot\begin{pmatrix}6\\3\\-12\end{pmatrix} = 0$$

(iii) Area of triange *OAC* = $\left|\overrightarrow{OA}\times\overrightarrow{OM}\right|$

$$= \left|\begin{pmatrix}-3\\0\\9\end{pmatrix}\times\begin{pmatrix}1\\2\\1\end{pmatrix}\right|$$

OR

Area of triangle =

$$\left|\begin{pmatrix}1\\2\\1\end{pmatrix}\right|\times\left|\begin{pmatrix}4\\2\\-8\end{pmatrix}\right|$$

$$= \sqrt{6}\times\sqrt{84}$$

	$= \left\| \begin{pmatrix} -18 \\ 12 \\ -6 \end{pmatrix} \right\|$ $= 22.4$ units^2 (or $6\sqrt{14}$ units^2)

12. Since $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and *A*, *B* and *C* are collinear,

$\therefore k = 2$

$$\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$$
$$= \mathbf{b} + 2\mathbf{b} - 2\mathbf{a}$$
$$= 3\mathbf{b} - 2\mathbf{a}$$

$$\frac{3}{4}|\mathbf{a}| = \left| \mathbf{b}.\frac{\mathbf{a}}{|\mathbf{a}|} \right|$$

$$\frac{3}{4}|\mathbf{a}|^2 = |4|$$

$$|\mathbf{a}|^2 = \frac{16}{3}$$

$$|\mathbf{a}| = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

$$\frac{3}{4}\mathbf{a} = \left(\mathbf{b} \cdot \hat{\mathbf{a}}\right)\hat{\mathbf{a}} = \frac{(\mathbf{b} \cdot \mathbf{a})(\mathbf{a})}{|\mathbf{a}|^2}$$

Alternatively,

$$\Rightarrow \frac{3}{4}|\mathbf{a}| = \frac{4|\mathbf{a}|}{|\mathbf{a}|^2}$$

$$\frac{3}{4}|\mathbf{a}|^2 = |4|$$

$$|\mathbf{a}|^2 = \frac{16}{3}$$

$$|\mathbf{a}| = \frac{4}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$$

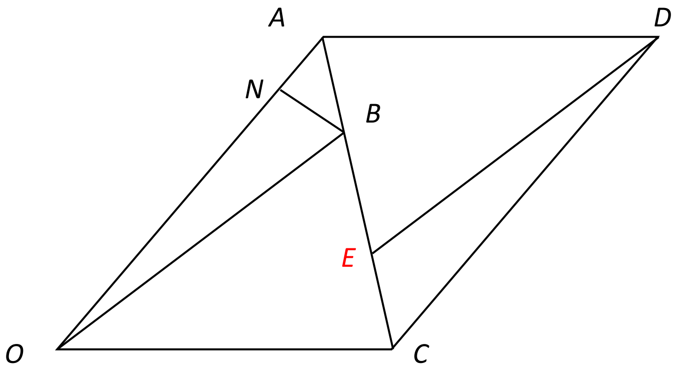
$$|\mathrm{b}| = \sqrt{2^2 + 2^2 + 2^2} = 2\sqrt{3}$$

$$\cos\theta = \frac{\mathrm{a.b}}{|\mathrm{a}||\mathrm{b}|}$$

$$= \frac{4}{\left(\frac{4\sqrt{3}}{3}\right)\left(2\sqrt{3}\right)}$$

$$= \frac{1}{2}$$

$$\theta = \frac{\pi}{3} \text{ or } 60^0$$



$$\overrightarrow{DE} = \frac{2\overrightarrow{DC} + \overrightarrow{DA}}{3}$$

$$= \frac{2(-\mathbf{a}) + (2\mathbf{a} - 3\mathbf{b})}{3}$$

$$= -\mathbf{b}$$

$$= \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$$

Alternatively,

$$\overrightarrow{AE} = \frac{2}{3}\overrightarrow{AE} = \frac{2}{3}(3\mathbf{b} - 2\mathbf{a} - \mathbf{a}) = 2\mathbf{b} - 2\mathbf{a}$$

$$\overrightarrow{DE} = \overrightarrow{DA} + \overrightarrow{AE} = \overrightarrow{CO} + \overrightarrow{AE}$$

$$= 2\mathbf{a} - 3\mathbf{b} + 2\mathbf{b} - 2\mathbf{a}$$

$$= -\mathbf{b}$$

$$= \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$$