Note Title 23/11/2015

2015 'A' Level H2 Maths P1 Solutions

1. System of Linear Equations / Graphing Techniques

(i) $y = \frac{a}{x^2} + bx + c \Rightarrow \frac{dy}{dx} = -2ax^{-1} + b$

At $(1.6, -2.4)$: $-2.4 = \frac{a}{1.6^2} + b(1.6) + c$ —— (1)

" $(-0.7, 3.6)$: $3.6 = \frac{a}{(-0.7)^2} + b(-0.7) + c$ —— (2)

When $x = 1$, $\frac{dy}{dx} = 2$, $2 = -2a + b$ —— (3)

Using GC, $a = -3.593$, $b = -5.187$, $c = 7.303$

(ii) $y = \frac{-3.593}{x^2} - 5.187x + 7.303$

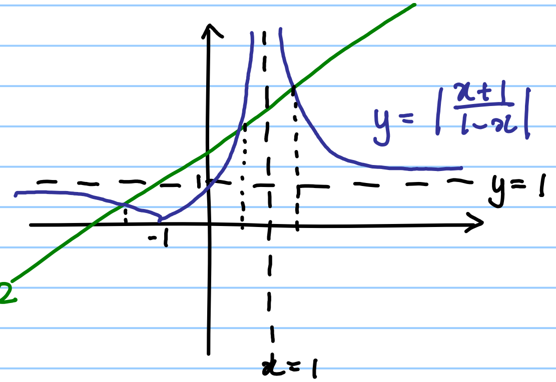
When $y = 0$, $-\frac{3.593}{x^2} - 5.187x + 7.303 = 0$

Using GC, $x = -0.589$

(iii) The other asymptote is $y = -5.187x + 7.303$

2. Graphing Techniques / Inequalities

(i)

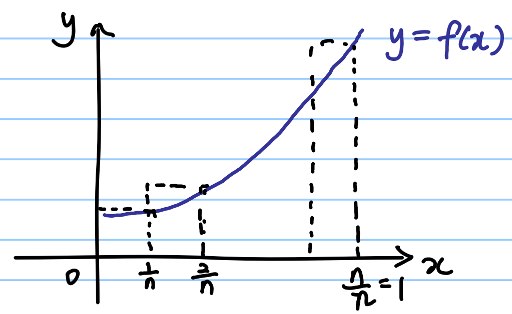


(ii) $\left|\frac{x+1}{1-x}\right| < x+2$

$\Rightarrow \quad -1.73 < x < 0.414 \quad \text{or} \quad x > 1.73$

3. Application of Integration

(i)



Sum of area of n rectangles

$= \frac{1}{n}\left\{f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right)\right\}$

When $n \to \infty$,

Sum of area of n rectangles

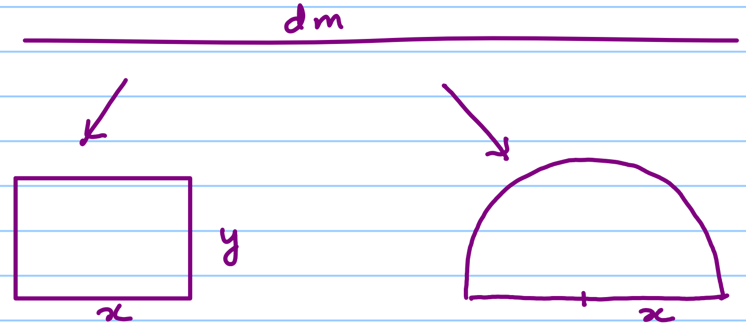
$\longrightarrow$ Actual area under curve

$\therefore \lim_{n\to\infty} \frac{1}{n}\left\{f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right)\right\} = \int_0^1 f(x)\,dx$

(ii) $\lim_{n\to\infty} \frac{1}{n}\left(\frac{\sqrt[3]{1} + \sqrt[3]{2} + \dots + \sqrt[3]{n}}{\sqrt[3]{n}}\right) = \lim_{n\to\infty} \frac{1}{n}\left(\sqrt[3]{\frac{1}{n}} + \sqrt[3]{\frac{2}{n}} + \dots + \sqrt[3]{\frac{n}{n}}\right)$

$= \int_0^1 \sqrt[3]{x}\,dx = \left[\frac{x^{4/3}}{4/3}\right]_0^1 = \frac{3}{4}$

4. Application of Differentiation



$$d = 2(x+y) + \pi x + 2x \Rightarrow y = \frac{d - \pi x - 4x}{2}$$

$$A = xy + \tfrac{1}{2}\pi x^2$$

$$= x\left(\frac{d - \pi x - 4x}{2}\right) + \tfrac{1}{2}\pi x^2$$

$$= \frac{d}{2}x - 2x^2$$

$$\frac{dA}{dx} = \frac{d}{2} - 4x$$

When A is max, $\frac{dA}{dx} = 0$

$$\frac{d}{2} - 4x = 0$$

$$x = \frac{d}{8}$$

$$\frac{d^2A}{dx^2} = -4 < 0 \Rightarrow A \text{ is max when } x = \frac{d}{8}$$

$$\therefore \text{Max } A = \frac{d}{2}\left(\frac{d}{8}\right) - 2\left(\frac{d}{8}\right)^2$$

$$= \frac{1}{32}d^2$$

5. Graphing Techniques / Transformation of Curves

(i)

$y = x^2$

$\downarrow$ $x \to x-3$

Translation along x-axis by 3 units

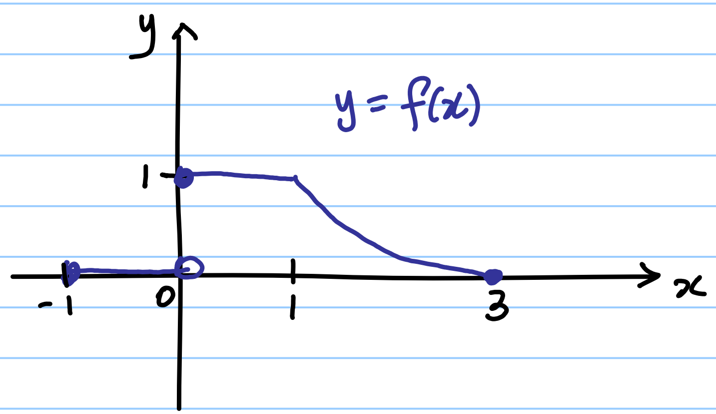
$y = (x-3)^2$

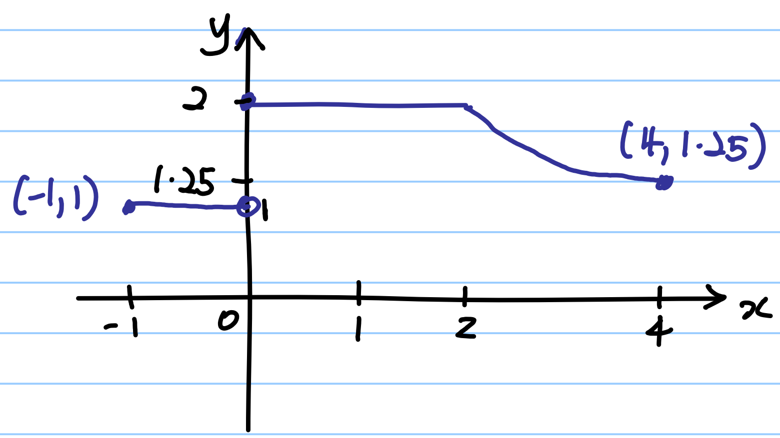
$\downarrow$ $y \to 4y$

Scaling along y-axis by factor $\frac{1}{4}$

$y = \frac{1}{4}(x-3)^2$

(ii)





When $x = 2$, $y = 1 + f(1)$

$= 1 + 1 = 2$

" $x = 4$, $y = 1 + f(2)$

$= 1 + \frac{1}{4}(2-3)^2$

$= 1.25$

6. Maclaurin's Series / Binomial Expansion

(i) $\ln(1+2x) = (2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} + \dots$

$$\approx 2x - 2x^2 + \frac{8x^3}{3}$$

(ii) $ax(1+bx)^c = ax\left(1 + c(bx) + \frac{c(c-1)}{2!}(bx)^2 + \frac{c(c-1)(c-2)}{3!}(bx)^3 + \dots\right)$

$$= ax + abc\,x^2 + \frac{ac(c-1)}{2}b^2x^3 + \frac{ac(c-1)(c-2)}{6}b^3x^4 + \dots$$

By comparing coefficients of both expansions,

$$a = 2$$

$$abc = -2 \Rightarrow b = \frac{-1}{c}$$

$$\frac{ac(c-1)}{2}b^2 = \frac{8}{3} \Rightarrow \frac{\cancel{2}c(c-1)}{\cancel{2}}\left(\frac{-1}{c}\right)^2 = \frac{8}{3}$$

$$\Rightarrow \frac{c-1}{c} = \frac{8}{3}$$

$$\Rightarrow 3c - 3 = 8c$$

$$c = \frac{-3}{5}$$

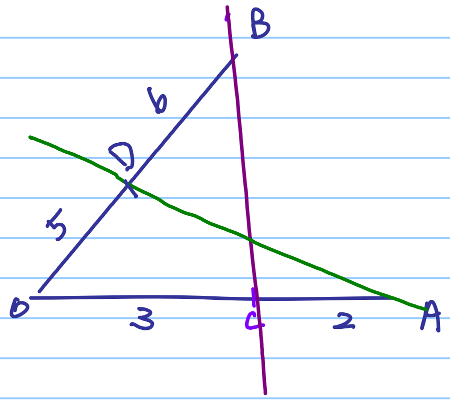
$$\therefore a = 2, \quad b = \frac{5}{3}, \quad c = -\frac{3}{5}$$

$\therefore$ Coefficient of $x^4 = \frac{2\left(-\frac{3}{5}\right)\left(-\frac{8}{5}\right)\left(-\frac{13}{5}\right)}{6}\left(\frac{5}{3}\right)^3$

$$= -\frac{104}{27}$$

7. Vectors

(i)



$\vec{OC} = \frac{3}{5}\underset{\sim}{a}$

$\vec{OD} = \frac{5}{11}\underset{\sim}{b}$

(ii) $l_{BC}: \underset{\sim}{r} = \vec{OB} + \lambda\vec{BC}$

$= \underset{\sim}{b} + \lambda\left(\frac{3}{5}\underset{\sim}{a} - \underset{\sim}{b}\right)$

$\therefore \underset{\sim}{r} = \frac{3}{5}\lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}$ (shown)

$l_{AD}: \underset{\sim}{r} = \vec{OA} + \mu\vec{AD}$

$= \underset{\sim}{a} + \mu\left(\frac{5}{11}\underset{\sim}{b} - \underset{\sim}{a}\right)$

$\therefore \underset{\sim}{r} = (1-\mu)\underset{\sim}{a} + \frac{5}{11}\mu\underset{\sim}{b}$

(iii) Let $\frac{3}{5}\lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b} = \frac{5}{11}\mu\underset{\sim}{b} + (1-\mu)\underset{\sim}{a}$

Comparing coefficients on both sides,

$\left.\begin{array}{l} \frac{3}{5}\lambda = 1-\mu \Rightarrow \frac{3}{5}\lambda + \mu = 1 \\ 1-\lambda = \frac{5}{11}\mu \Rightarrow \lambda + \frac{5}{11}\mu = 1 \end{array}\right\}$ Using GC, $\lambda = \frac{3}{4}$, $\mu = \frac{11}{20}$

$\therefore \vec{OE} = \frac{3}{5}\left(\frac{3}{4}\right)\underset{\sim}{a} + \left(1-\frac{3}{4}\right)\underset{\sim}{b}$

$= \frac{9}{20}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}$

$\vec{AE} = \frac{9}{20}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b} - \underset{\sim}{a} = -\frac{11}{20}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}$

$\vec{ED} = \frac{5}{11}\underset{\sim}{b} - \left(\frac{9}{20}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}\right) = -\frac{9}{20}\underset{\sim}{a} + \frac{9}{44}\underset{\sim}{b}$

$\frac{\vec{AE}}{\vec{ED}} = \frac{-\frac{11}{20}\underset{\sim}{a} + \frac{1}{4}\underset{\sim}{b}}{-\frac{9}{20}\underset{\sim}{a} + \frac{9}{44}\underset{\sim}{b}} = \frac{11\left(-\frac{1}{20}\underset{\sim}{a} + \frac{1}{44}\underset{\sim}{b}\right)}{9\left(-\frac{1}{20}\underset{\sim}{a} + \frac{1}{44}\underset{\sim}{b}\right)} = \frac{11}{9}$

$\therefore AE : ED = 11 : 9$

8. AP / GP

(i) Athlete A : AP $\quad a = T\,s,\ d = 2\,s$

$$S_{50} = \frac{50}{2}\left[2T + 49(2)\right] = 50(T+49)$$

$$1.5 \text{ hrs} \le S_{50} \le 1.75 \text{ hrs}$$

$$1.5 \times 60 \times 60 \le 50(T+49) \le 1.75 \times 60 \times 60$$

$$108 \le T+49 \le 126$$

$$59 \le T \le 77$$

$\therefore$ Set of values of $T = \{T \in \mathbb{R} : 59 \le T \le 77\}$

(ii) Athlete B : GP $\quad a = t,\ r = 1.02$

$$S_{50} = \frac{t(1.02^{50}-1)}{1.02-1} = 50(1.02^{50}-1)$$

$$1.5 \times 60 \times 60 \le 50t(1.02^{50}-1) \le 1.75 \times 60 \times 60$$

$$108 \le t(1.02^{50}-1) \le 126$$

$$63.845 \le t \le 74.486$$

$\therefore$ Set of values of $t = \{t \in \mathbb{R} : 63.9 \le t \le 74.4\}$

(iii) Athlete A : $T_{50} = 59 + 49(2) = 157$

" B : $T_{50} = 63.84(1.02)^{49} = 168.47$

$\therefore$ Difference in athletes' times $= 168.47 - 157 = 11.47$
$\approx 11\,s$ (nearest second)

9. Complex Numbers

(a) $\dfrac{w^2}{w^*} = \dfrac{(a+ib)^2}{a-ib} = \dfrac{(a^2-b^2)+2abi}{a-ib} \times \dfrac{a+ib}{a+ib}$

$= a(a^2-b^2) - 2ab^2 + i\left(b(a^2-b^2) + 2a^2b\right)$

$\dfrac{w^2}{w^*}$ is purely imaginary $\Rightarrow a(a^2-b^2) - 2ab^2 = 0$

$\Rightarrow a(a^2-3b^2) = 0$

$a = 0$ (rejected $\because a \neq 0$) or $a^2 = 3b^2$

$\Rightarrow b = \pm \dfrac{a}{\sqrt{6}}$

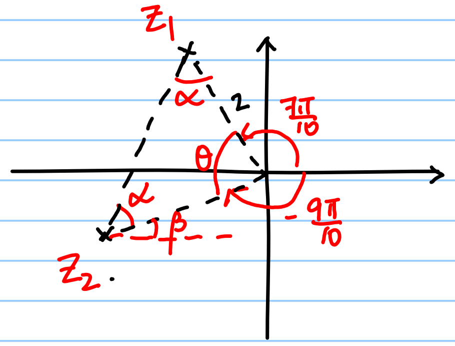
$\therefore w = a + i\left(\pm \dfrac{a}{\sqrt{6}}\right) = a \pm i\left(\dfrac{a}{\sqrt{3}}\right)$

(b) (i) $z^5 = -32i = 32\left(e^{i(-\frac{\pi}{2})}\right) = 32e^{i(-\frac{\pi}{2}+2k\pi)}$

$z = 2e^{i(-\frac{\pi}{10}+\frac{2k\pi}{5})}$, $k = 0, \pm 1, \pm 2$

$= 2e^{i(-\frac{\pi}{10})}, \; 2e^{i(\frac{3\pi}{10})}, \; 2e^{i(-\frac{\pi}{2})}, \; 2e^{i(\frac{7\pi}{10})}, \; 2e^{i(-\frac{9\pi}{10})}$

(ii)



$z_1 = 2e^{i(\frac{7\pi}{10})}$, $z_2 = 2e^{i(-\frac{9\pi}{10})}$

$\theta = 2\pi - \frac{7\pi}{10} - \frac{9\pi}{10} = \frac{2\pi}{5}$

$\alpha = \dfrac{\pi - \frac{2\pi}{5}}{2} = \dfrac{3\pi}{10}$

$\beta = \pi - \frac{9\pi}{10} = \frac{\pi}{10}$

$\arg(z_1 - z_2) = \alpha + \beta = \frac{3\pi}{10} + \frac{\pi}{10} = \frac{2\pi}{5}$

$|z_1 - z_2|^2 = 2^2 + 2^2 - 2(2)(2)\cos\frac{2\pi}{5}$

$= 8 - 8\cos\frac{2\pi}{5} = 8 - 8\left(1 - 2\sin^2\left(\frac{\pi}{5}\right)\right)$

$= 16\sin^2\left(\frac{\pi}{5}\right)$

$\therefore |z_1 - z_2| = 4\sin\frac{\pi}{5}$ (shown)

10. Applications of Integration

(i) At P, $\sin x = \cos x \Rightarrow x = \frac{\pi}{4}$

$$A_1 = \int_0^{\frac{\pi}{4}} \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$$

$$= \left[-\cos x\right]_0^{\frac{\pi}{4}} + \left[\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \left[-\frac{1}{\sqrt{2}} + 1\right] + \left[1 - \frac{1}{\sqrt{2}}\right]$$

$$= 2 - \frac{2}{\sqrt{2}} = 2 - \sqrt{2}$$

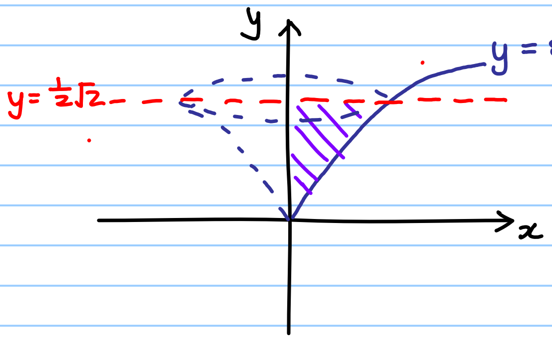
$$A_1 + A_2 = \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$$

$$A_2 = 1 - (2 - \sqrt{2}) = \sqrt{2} - 1$$

$$\frac{A_1}{A_2} = \frac{2 - \sqrt{2}}{\sqrt{2} - 1} = \frac{\sqrt{2}(\sqrt{2} - 1)}{\sqrt{2} - 1} = \sqrt{2} \text{ (shown)}$$

(ii)



$y = \sin x \Rightarrow x = \sin^{-1} y$

$$\text{Required Volume} = \pi \int_0^{\frac{1}{2}\sqrt{2}} x^2 \, dy$$

$$= \pi \int_0^{\frac{1}{2}\sqrt{2}} (\sin^{-1} y)^2 \, dy \quad \text{(shown)}$$

$$= \pi \int_0^{\frac{\pi}{4}} u^2 \cos u \, du$$

$$= \pi \left\{ \left[u^2 \sin u\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sin u \,(2u) \, du \right.$$

$$= \pi \left\{ \frac{\pi^2}{16}\left(\frac{1}{\sqrt{2}}\right) - \left[2u(-\cos u)\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} (-\cos u)(2) \, du \right.$$

$$= \pi \left\{ \frac{\pi^2}{16\sqrt{2}} - \left[\frac{\pi}{2}\left(-\frac{1}{\sqrt{2}}\right) - 0\right] - 2\left[\sin u\right]_0^{\frac{\pi}{4}} \right\}$$

$$= \pi \left\{ \frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}} - 2\left[\frac{1}{\sqrt{2}} - 0\right] \right\}$$

$$= \pi \left(\frac{\sqrt{2}\pi^2}{32} + \frac{\sqrt{2}\pi}{4} - \sqrt{2} \right)$$

$y = \sin u \Rightarrow \frac{dy}{du} = \cos u$

When $y = \frac{1}{2}\sqrt{2}$, $u = \frac{\pi}{4}$

" $y = 0$, $u = 0$

11. Applications of Differentiation + Integration

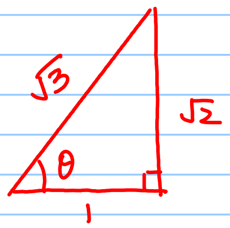
(i) $x = \sin^3\theta \Rightarrow \frac{dx}{d\theta} = 3\sin^2\theta\cos\theta$

$y = 3\sin^2\theta\cos\theta \Rightarrow \frac{dy}{d\theta} = 3\sin^2\theta(-\sin\theta) + \cos\theta(6\sin\theta\cos\theta)$
$= -3\sin^3\theta + 6\sin\theta\cos^2\theta$

$$\frac{dy}{dx} = \frac{-3\sin^3\theta + 6\sin\theta\cos^2\theta}{3\sin^2\theta\cos\theta} = \frac{-\sin\theta}{\cos\theta} + \frac{2\cos\theta}{\sin\theta}$$
$$= 2\cot\theta - \tan\theta \quad \text{(shown)}$$

(ii) At turning pt, $\frac{dy}{dx} = 0 \Rightarrow 2\cot\theta - \tan\theta = 0$

$\frac{2}{\tan\theta} = \tan\theta \quad \therefore \tan\theta = \sqrt{2}$



$\cos\theta = \frac{1}{\sqrt{3}}, \quad \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$

$x = \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^3 = \frac{2}{3}\sqrt{\frac{2}{3}} = \frac{2}{9}\sqrt{6}$

$y = 3\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{3}\sqrt{3}$

$\therefore$ Turning point $= \left(\frac{2}{9}\sqrt{6}, \frac{2}{3}\sqrt{3}\right)$

$\frac{d^2y}{dx^2} = 2(-\operatorname{cosec}^2\theta - \sec^2\theta) < 0$

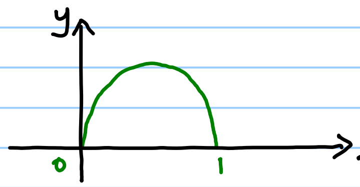
$\Rightarrow \therefore \left(\left(\frac{2}{3}\right)^{3/2}, \frac{2}{\sqrt{3}}\right)$ is a maximum point.

(iii) Area $= \int_0^1 y\,dx$

$= \int_0^{\frac{\pi}{2}} (3\sin^2\theta\cos\theta)(3\sin^2\theta\cos\theta\,d\theta)$

$= \int_0^{\frac{\pi}{2}} 9\sin^4\theta\cos^3\theta\,d\theta$ (shown)

$= 0.884$



When $y = \sin^2\theta\cos\theta = 0$,
$\sin\theta = 0$ or $\cos\theta = 0$
$\theta = 0$ or $\theta = \frac{\pi}{2}$
$x = 0$ or $x = 1$

(iv) $y = ax \Rightarrow 3\sin^2\theta\cos\theta = a\sin^3\theta$

$\sin^2\theta(3\cos\theta - a\sin\theta) = 0$

$\sin^2\theta = 0$ or $3\cos\theta - a\sin\theta = 0$

At P, $3\cos\theta = a\sin\theta \quad \therefore \tan\theta = \frac{3}{a}$

At stationary point, $\tan\theta = \sqrt{2} \Rightarrow \frac{3}{a} = \sqrt{2}$

$\therefore a = \frac{3}{\sqrt{2}} = \frac{3\sqrt{2}}{2}$