2015 'A' Level H2 Maths P2 Solutions

1. Differential Equations

(i) $\frac{dh}{dt} = \frac{1}{10}\sqrt{16-\frac{1}{2}h}$

At max. height, $\frac{dh}{dt} = 0 \Rightarrow \frac{1}{10}\sqrt{16-\frac{1}{2}h} = 0$

$16-\frac{1}{2}h = 0 \quad \therefore \text{ max } h = 32 \text{ m}$

(ii) $\int \frac{1}{\sqrt{16-\frac{1}{2}h}}\, dh = \frac{1}{10}\int dt$

$\frac{(16-\frac{1}{2}h)^{\frac{1}{2}}}{(\frac{1}{2})(-\frac{1}{2})} = \frac{1}{10}t + c$

When $t=0$, $h=0$,

$\frac{4}{(-\frac{1}{4})} = c \Rightarrow c = -16$

$-4\sqrt{16-\frac{1}{2}h} = \frac{1}{10}t - 16$

$\therefore t = 10\left(16 - 4\sqrt{16-\frac{1}{2}h}\right) = 40\left(4-\sqrt{16-\frac{1}{2}h}\right)$

When $h = \frac{1}{2}(\text{max ht}) = 16$,

$t = 40\left(4 - \sqrt{16-8}\right)$

$= 46.9$ yrs

2. Vectors

(i) $L : \underset{\sim}{r} = \begin{pmatrix}1\\-2\\-4\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-6\end{pmatrix}, \quad \lambda \in \mathbb{R}$

Acute $\measuredangle$ between L and x-axis

$$= \cos^{-1}\frac{\left|\begin{pmatrix}2\\3\\-6\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\right|}{\sqrt{4+9+36}\sqrt{1}} = 73.4^\circ$$

(ii) $\vec{OP} = \begin{pmatrix}2\\5\\-6\end{pmatrix}$

Let X be a point on L,

then $\vec{OX} = \begin{pmatrix}1\\-2\\-4\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-6\end{pmatrix}$ for some $\lambda \in \mathbb{R}$

$$\vec{XP} = \begin{pmatrix}2\\5\\-6\end{pmatrix} - \begin{pmatrix}1+2\lambda\\-2+3\lambda\\-4-6\lambda\end{pmatrix} = \begin{pmatrix}1-2\lambda\\7-3\lambda\\-2+6\lambda\end{pmatrix}$$

$$|\vec{XP}| = \sqrt{(1-2\lambda)^2+(7-3\lambda)^2+(-2+6\lambda)^2} = \sqrt{33}$$

$$(1-4\lambda+4\lambda^2)+(49-42\lambda+9\lambda^2)+(4-24\lambda+36\lambda^2) = 33$$

$$49\lambda^2 - 70\lambda + 21 = 0$$

$$7(7\lambda-3)(\lambda-1) = 0$$

$$\lambda = \frac{3}{7} \quad \text{or} \quad \lambda = 1$$

$\therefore \vec{OX} = \begin{pmatrix}1\\-2\\-4\end{pmatrix} + \frac{3}{7}\begin{pmatrix}2\\3\\-6\end{pmatrix} = \begin{pmatrix}13/7\\-5/7\\-46/7\end{pmatrix}$

or $\vec{OX} = \begin{pmatrix}1\\-2\\-4\end{pmatrix} + \begin{pmatrix}2\\3\\-6\end{pmatrix} = \begin{pmatrix}3\\1\\-10\end{pmatrix}$

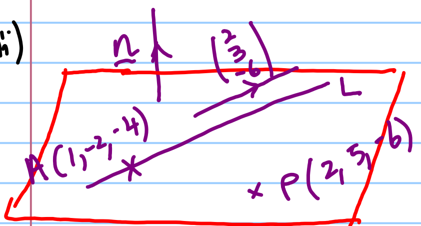
$\therefore X = (13/7, -5/7, -46/7)$ or $(3, 1, -10)$

Position vector of point on L closest to P

$$= \frac{1}{2}\left[\begin{pmatrix}13/7\\-5/7\\-46/7\end{pmatrix} + \begin{pmatrix}3\\1\\-10\end{pmatrix}\right] = \begin{pmatrix}17/7\\1/7\\-58/7\end{pmatrix}$$

$\therefore$ Point on L closest to $P = (17/7, 1/7, -58/7)$

(iii)



$$\vec{AP} = \begin{pmatrix}2\\5\\-6\end{pmatrix} - \begin{pmatrix}1\\-2\\-4\end{pmatrix} = \begin{pmatrix}1\\7\\-2\end{pmatrix}$$

$$\underset{\sim}{n} = \begin{pmatrix}1\\7\\-2\end{pmatrix} \times \begin{pmatrix}2\\3\\-6\end{pmatrix} = \begin{pmatrix}-36\\2\\-11\end{pmatrix}$$

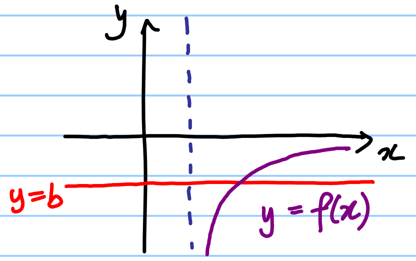
$\therefore$ Equation of plane is $\underset{\sim}{r}\cdot\begin{pmatrix}-36\\2\\-11\end{pmatrix} = \begin{pmatrix}2\\5\\-6\end{pmatrix}\cdot\begin{pmatrix}-36\\2\\-11\end{pmatrix} = 4$

$$\therefore -36x + 2y - 11z = 4$$

3. Functions

(a) $f : x \longrightarrow \dfrac{1}{1-x^2}, \quad x \in \mathbb{R}, \quad x > 1$

(i)



$\because$ Every horizontal line $y=b$, $b \in (-\infty, 0)$ cuts the graph at exactly 1 point, f is 1-1.

$\therefore$ f^{-1} exists

i.e. f has an inverse.

(ii) Let $y = \dfrac{1}{1-x^2} \Rightarrow 1 - x^2 = \dfrac{1}{y}$

$$x^2 = 1 - \frac{1}{y}$$

$$x = \sqrt{1-\frac{1}{y}} \quad \text{or} \quad -\sqrt{1-\frac{1}{y}}$$

(rejected $\because x > 1$)

$\therefore f^{-1}(x) = \sqrt{1 - \frac{1}{x}}$

and $D_{f^{-1}} = R_f = (-\infty, 0)$

(b) $g : x \longrightarrow \dfrac{2+x}{1-x^2}, \quad x \in \mathbb{R}, \quad x \neq \pm 1$

Let $y = \dfrac{2+x}{1-x^2} \Rightarrow (1-x^2)y = 2 + x$

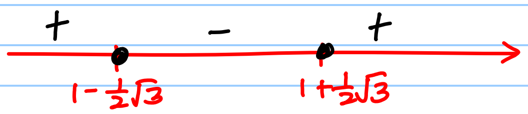
$$yx^2 + x + 2 - y = 0$$

For graph of $g(x)$ to exist, $b^2 - 4ac \geqslant 0$

$$1 - 4(y)(2-y) \geqslant 0$$

$$4y^2 - 8y + 1 \geqslant 0$$

When $4y^2 - 8y + 1 = 0$, $y = \dfrac{8 \pm \sqrt{64 - 4(4)}}{8} = 1 \pm \frac{1}{2}\sqrt{3}$



$\therefore y \leqslant 1 - \frac{1}{2}\sqrt{3}$ or $y \geqslant 1 + \frac{1}{2}\sqrt{3}$

$\therefore R_g = (-\infty, 1 - \frac{1}{2}\sqrt{3}] \cup [1 + \frac{1}{2}\sqrt{3}, \infty)$

4. Mathematical Induction / Σ Notation + Method of Difference

(a) Let P_n be the statement

$$\sum_{r=1}^{n} r(r+2)(r+5) = \frac{1}{12}n(n+1)(3n^2+31n+74) \quad \forall n \in \mathbb{Z}^+$$

When $n=1$, $\left.\begin{array}{l} LHS = (1)(3)(6) = 18 \\ RHS = \frac{1}{12}(2)(3+31+74) = 18 \end{array}\right\}$ $\because LHS = RHS$, $\therefore P_1$ is true

Assume P_k is true for some $k \in \mathbb{Z}^+$

i.e. $\sum_{r=1}^{k} r(r+2)(r+5) = \frac{1}{12}k(k+1)(3k^2+31k+74)$

To prove P_{k+1} is true i.e. $\sum_{r=1}^{k+1} r(r+2)(r+5) = \frac{1}{12}(k+1)(k+2)(3(k+1)^2+31(k+1)+74)$

$$= \frac{1}{12}(k+1)(k+2)(3k^2+37k+108)$$

$$\begin{aligned}
LHS &= \sum_{r=1}^{k+1} r(r+2)(r+5) \\
&= \sum_{r=1}^{k} r(r+2)(r+5) + (k+1)(k+3)(k+1+5) \\
&= \frac{1}{12}k(k+1)(3k^2+31k+74) + (k+1)(k+3)(k+6) \\
&= \frac{1}{12}(k+1)\left[k(3k^2+31k+74) + 12(k+3)(k+6)\right] \\
&= \frac{1}{12}(k+1)\left[3k^3+31k^2+74k+12(k^2+9k+18)\right] \\
&= \frac{1}{12}(k+1)\left[3k^3+43k^2+182k+216\right] \\
&= \frac{1}{12}(k+1)(k+2)(3k^2+37k+108) \\
&= RHS
\end{aligned}$$

$\therefore P_k$ is true $\Rightarrow P_{k+1}$ is true

$\because P_1$ is true and P_k is true $\Rightarrow P_{k+1}$ is true,

$\therefore$ By Mathematical Induction, P_n is true $\forall n \in \mathbb{Z}^+$.

(b)(i) $\frac{2}{4r^2+8r+3} = \frac{2}{(2r+3)(2r+1)}$

By cover-up rule,

$$= \frac{A}{2r+3} + \frac{B}{2r+1}$$

$$A = \frac{2}{2(-\frac{3}{2})+1} = -1$$

$$= \frac{-1}{2r+3} + \frac{1}{2r+1}$$

$$B = \frac{2}{2(-\frac{1}{2})+3} = 1$$

(ii) $S_n = \sum_{r=1}^{n} \frac{2}{4r^2+8r+3} = \sum_{r=1}^{n}\left(\frac{1}{2r+1} - \frac{1}{2r+3}\right)$

$$\left.\begin{aligned} = {} & \frac{1}{3} - \cancel{\frac{1}{5}} \\ & + \cancel{\frac{1}{5}} - \cancel{\frac{1}{7}} \\ & + \cancel{\frac{1}{7}} - \cancel{\frac{1}{9}} \\ & \quad \vdots \\ & + \cancel{\frac{1}{2n-1}} - \cancel{\frac{1}{2n+1}} \\ & + \cancel{\frac{1}{2n+1}} - \frac{1}{2n+3} \end{aligned}\right\} \therefore S_n = \frac{1}{3} - \frac{1}{2n+3}$$

(iii) $|S_n - S_\infty| < 10^{-3}$

$$\left|\left(\frac{1}{3} - \frac{1}{2n+3}\right) - \frac{1}{3}\right| < 10^{-3} \Rightarrow \frac{1}{2n+3} < 10^{-3}$$

$$2n+3 > 1000$$

$$n > 498.5$$

$\therefore$ least $n = 499$

5. Sampling Methods

(i) Manager would not be able to use stratified sampling as sampling frame which is the list of customers of different ages is not available.

(ii) Divide the customers into different age groups such as below 20 yrs old, 21 – 40 yrs old, 41 – 60 yrs old and above 60 yrs old. Decide on the no. of customers to be selected from each age group. Station at the entrance of supermarket and survey the required no. of customers that first enter the supermarket from each age group.

(iii) Quota sample would not be representative as the sample selected is biased and not proportional to the sample size of each age group.

6. Binomial Distribution and Approximation.

(i) Let X be no. of red sweets in a small packet of Droppers.

$X \sim B(10, 0.25)$

$P(X \geqslant 4) = 1 - P(X \leqslant 3) = 0.224$

(ii) Let Y be no. of red sweets in large packet of Droppers

$Y \sim B(100, 0.25)$

$\because$ $n = 100$ is large, $np = 100(0.25) = 25 > 5$,
$nq = 100(0.75) = 75 > 5$,

$Y \sim N(25, 18.75)$ approx.

$P(Y \geqslant 30) \overset{c.c.}{\equiv} P(X \geqslant 29.5) = 0.14935$
≈ 0.149

(iii) Let W be no. of packets of Droppers with $\geqslant 30$ red sweets

$W \sim B(15, 0.14935)$

$P(W \leqslant 3) = 0.825$

7. Poisson Distribution

(i) 2 Assumptions :

The average no. of errors per page is constant for each page

The errors occurred is independent of each other in each page

(ii) Let X be no of errors in 6 pages

$X \sim Po(1.3 \times 6)$

$P(X > 10) = 1 - P(X \leq 10) = 0.165$

(iii) Let y be no. of errors in n pages

$X \sim Po(1.3n)$

$P(X < 2) < 0.05$

$e^{-1.3n} + e^{-1.3n}(1.3n) < 0.05$

$e^{-1.3n}(1 + 1.3n) < 0.05$

n	$e^{-1.3n}(1+1.3n)$	
3	0.09919	> 0.05
4	0.0342	< 0.05
5	0.01128	< 0.05

Using GC, $n \geq 4$ $\therefore$ least $n = 4$.

8. Hypothesis Testing

0.8 1.00 0.82 0.85 0.93 0.96 0.81 0.89

Using GC, unbiased estimate of $\mu = \bar{x} = 0.8825$

" " " $\sigma^2 = s^2 = (0.074785)^2 = 0.00559$

Let X be the mass of pineapples

$H_0 : \mu = 0.9$

$H_1 : \mu < 0.9$

$\because$ $n = 8$ is small and σ is unknown, we use t-test.

Test Statistic, $T = \dfrac{\bar{X} - 0.9}{S/\sqrt{8}} \sim T(7)$

Using GC, $p = 0.265$, $t = -0.662$

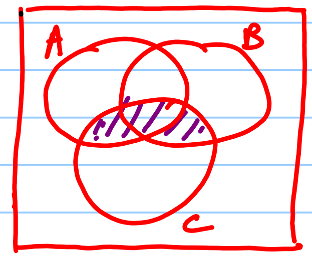
$\because$ $p = 0.265 > 0.1$, we do not reject H_0 and conclude that there's insufficient evidence at 10% level of significance to doubt the stall owner's claim.

9. **Probability**

$P(A) = 0.45$, $P(B) = 0.4$, $P(C) = 0.3$, $P(A \cap B \cap C) = 0.1$

(i) $P(B|A) = P(B)$ ($\because$ A & B are independent)
$= 0.4$

(ii) $P(A' \cap B' \cap C') = 1 - P(A \cup B \cup C)$
$= 1 - [P(A \cup B) + P(C) - P((A \cup B) \cap C)]$

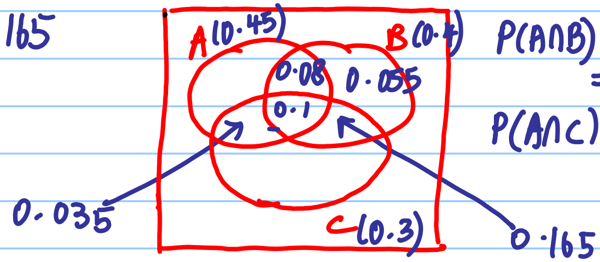


$\boxed{///}$ $(A \cup B) \cap C$

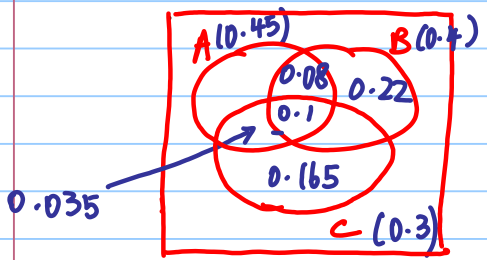
$P(A \cup B) = 0.45 + 0.4 - (0.45)(0.4) = 0.67$
$P((A \cup B) \cap C) = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$
$= (0.45)(0.3) + (0.4)(0.3) - 0.1$
$= 0.135 + 0.12 - 0.1 = 0.155$

$P(A' \cap B' \cap C') = 1 - [0.67 + 0.3 - 0.155] = 0.185$

(iii) Least $P(A \cup B \cup C) = 0.45 + 0.055 + 0.165$
$= 0.67$
Greatest $P(A' \cap B' \cap C') = 1 - 0.67$
$= 0.33$



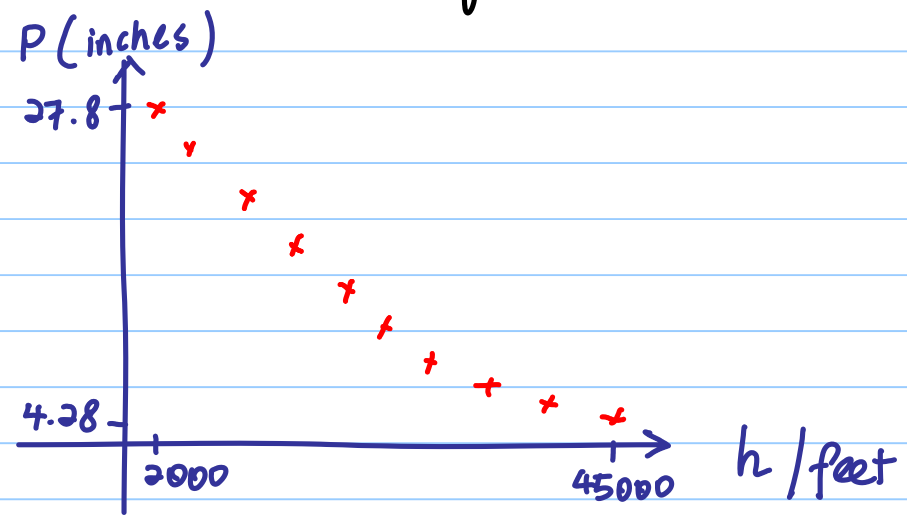
$P(A \cap B) = (0.45)(0.4)$
$= 0.18$
$P(A \cap C) = (0.45)(0.3)$
$= 0.135$



Greatest $P(A \cup B \cup C) = 0.45 + 0.22 + 0.165$
$= 0.835$
Least $P(A' \cap B' \cap C') = 1 - 0.835$
$= 0.165$

10. Correlation and Regression

(i)



(ii) (a) $r = -0.9807$
(b) $r = -0.9748$
(c) $r = -0.9986$

(iii) (c) is the best model $\because r = -0.9986$ is closest to -1.
$\because$ h is the independent variable,
regression line of P on $\sqrt{h}$ is $P = 34.7889 - 0.14686\sqrt{h}$
$\therefore P = 34.8 - 0.147\sqrt{h}$

(iv) $\because 1\text{ m} = 3.28$ feet, $P = 34.8 - 0.14686\sqrt{3.28h}$
$= 34.8 - 0.266\sqrt{h}$

11. Permutations and Combinations

CABBAGES (C A B G E S, A B)

(i) No. of arrangements $= \dfrac{8!}{2!\,2!} = 10080$

(ii) No. of arrangements in which letters not in alphabetical order $= 10080 - 1 = 10079$

(iii) No. of arrangements with both A's together and both B's together $= 6! = 720$

(iv) No of arrangements with no two adjacent letters the same

$$= \underbrace{10080}_{\text{total}} - \left[\underbrace{\frac{7!}{2!}}_{\text{2 'A's together}} + \underbrace{\frac{7!}{2!}}_{\text{2 'B's together}} - \underbrace{6!}_{\text{both 'A's + 'B's together}} \right] = 5760$$

12. Normal Distribution

(i) Let A and B be masses of apples and pears respectively

$A \sim N(300, 20^2)$, $B \sim N(200, 15^2)$

$P(A_1 + \dots + A_5 > 1600) = 0.0127$

(ii) Let $T = A_1 + \dots + A_5 - (B_1 + \dots + B_8) \sim N(-100, 3800)$

$E(T) = 5(300) - 8(200) = -100$

$Var(T) = 5(20^2) + 8(15^2) = 3800$

$P(A_1 + \dots + A_5 > B_1 + \dots + B_8) = P(T > 0)$

$= 0.0524$

(iii) Let $S = 0.85(A_1 + \dots + A_5) + 0.9(B_1 + \dots + B_8) \sim N(2715, 2903)$

$E(S) = 0.85(5 \times 300) + 0.9(8 \times 200) = 2715$

$Var(S) = 0.85^2(5 \times 20^2) + 0.9^2(8 \times 15^2) = 2903$

$P(T < 2750) = 0.742$