GCE A Level 2016 Paper 1

1. Inequalities

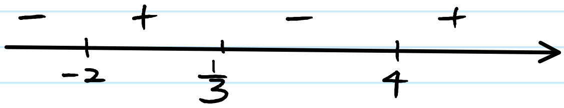
$$\frac{4x^2+4x-14}{x-4} - (x+3) = \frac{4x^2+4x-14-(x+3)(x-4)}{x-4}$$

$$= \frac{4x^2+4x-14-(x^2-x-12)}{x-4} = \frac{3x^2+5x-2}{x-4}$$

$$= \frac{(3x-1)(x+2)}{x-4} \#$$

$$\frac{4x^2+4x-14}{x-4} < x+3 \Rightarrow \frac{4x^2+4x-14}{x-4} - x+3 < 0$$

$$\frac{(3x-1)(x+2)}{x-4} < 0$$



$\therefore \quad x < -2 \quad \text{or} \quad \frac{1}{3} < x < 4 \#$

2. Differentiation & Applications

(i) $\frac{d}{dx}\left(2^{\cos x}\right)\Big|_{x=0} = 0$

$\frac{d}{dx}\left(2^{\cos x}\right)\Big|_{x=\frac{\pi}{2}} = -0.693147 \approx -0.693$

When $x=0$, $y = 2^{\cos 0} = 2$

Equation of tangent at $x=0$: $y-2=0$

$\therefore y = 2$

When $x=0$, $y = 2^{\cos\frac{\pi}{2}} = 1$

Equation of tangent at $x=\frac{\pi}{2}$: $y-1 = -0.693147\left(x-\frac{\pi}{2}\right)$

$y = -0.693147x + 2.08879 \Rightarrow \therefore y = -0.693x + 2.09$

When tangents meet, $-0.693x + 2.08879 = 2 \Rightarrow x = 0.128$

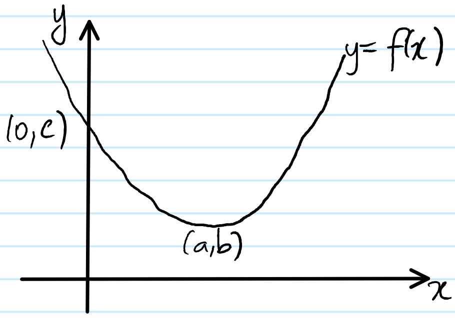
$\therefore$ pt of intersection $= (0.128, 2) \#$

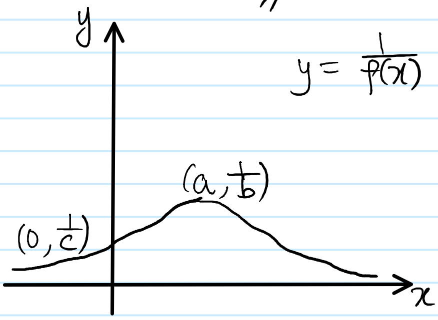
Transformations of Curves

3. $y = x^4 \longrightarrow y = f(x) = k(x-l)^4 + m$

$(0,0) \longrightarrow (a,b) \Rightarrow l = a, \quad m = b$

At $(0,c)$, $\quad c = k(-a)^4 + b \Rightarrow k = \dfrac{c-b}{a^4}$ #





4. AP / GP

(i) AP : T_4, T_9, T_{12}

GP : T_5, T_8, T_{15}

$a + 3d = br^4$ —— (1)

$a + 8d = br^7$ —— (2)

$a + 11d = br^{14}$ —— (3)

(2) − (1) : $5d = br^7 - br^4 = br^4(r^3 - 1)$ —— (4)

(3) − (2) : $3d = br^{14} - br^7 = br^7(r^7 - 1)$ —— (5)

$\dfrac{(4)}{(5)}$: $\dfrac{5}{3} = \dfrac{r^3 - 1}{r^3(r^7 - 1)}$

$5(r^{10} - r^3) = 3r^3 - 1$

$5r^{10} - 8r^3 + 3 = 0$ (shown)

Using GC, $\quad r = 0.740449 \simeq 0.74$ # (2dp)

(ii) Sum of terms after nth term $= S_\infty - S_n$

$= \dfrac{b}{1 - 0.740449} - \dfrac{b(1 - 0.740449^n)}{1 - 0.740449}$

$= \dfrac{b(0.74^n)}{0.26}$ #

Vectors (I)

5. $\underset{\sim}{u} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad \underset{\sim}{v} = \begin{pmatrix} a \\ 0 \\ b \end{pmatrix}$

(i) $(\underset{\sim}{u}+\underset{\sim}{v}) \times (\underset{\sim}{u}-\underset{\sim}{v}) = \underset{\sim}{u}\times\underset{\sim}{u} + \underset{\sim}{v}\times\underset{\sim}{u} - \underset{\sim}{u}\times\underset{\sim}{v} - \underset{\sim}{v}\times\underset{\sim}{v}$

$= 2\,\underset{\sim}{v}\times\underset{\sim}{u} = 2\begin{pmatrix} a \\ 0 \\ b \end{pmatrix} \times \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = 2\begin{pmatrix} b \\ 2b-2a \\ -a \end{pmatrix}$ #

(ii) $b = -a$

$(\underset{\sim}{u}+\underset{\sim}{v}) \times (\underset{\sim}{u}-\underset{\sim}{v}) = 2\begin{pmatrix} -a \\ -4a \\ -a \end{pmatrix} = -2a\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$ #

$(\underset{\sim}{u}+\underset{\sim}{v}) \times (\underset{\sim}{u}-\underset{\sim}{v})$ is a unit vector $\Rightarrow \left|-2a\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}\right| = 1$

$\left|-2a\sqrt{1+16+1}\right| = 1$

$|a| = \frac{1}{2\sqrt{18}} \Rightarrow a = \pm\frac{1}{6\sqrt{2}}$

(iii) $(\underset{\sim}{u}+\underset{\sim}{v}) \cdot (\underset{\sim}{u}-\underset{\sim}{v}) = 0 \Rightarrow \underset{\sim}{u}\cdot\underset{\sim}{u} - \underset{\sim}{v}\cdot\underset{\sim}{v} = 0$

$|\underset{\sim}{u}|^2 - |\underset{\sim}{v}|^2 = 0$

$|\underset{\sim}{v}|^2 = 2^2+1^2+2^2 = 9$

$|\underset{\sim}{v}| = 3$ or -3 (rejected $\because |\underset{\sim}{v}| > 0$)

M.I. / M.O.D.

6. Let P_n be the statement $\sum_{r=1}^{n} r(r^2+1) = \frac{1}{4}n(n+1)(n^2+n+2)$ for $\forall n \in \mathbb{Z}^+$

When $n=1$, $\left.\begin{aligned} LHS &= 1(1+1) = 2 \\ RHS &= \tfrac{1}{4}(1)(2)(4) = 2 \end{aligned}\right\}$ $\because$ LHS = RHS, P_1 is true.

Assume P_k is true for some $n \in \mathbb{Z}^+$

i.e. $\sum_{r=1}^{k} r(r^2+1) = \frac{1}{4}k(k+1)(k^2+k+2)$

To prove P_{k+1} is true i.e. $\sum_{r=1}^{k+1} r(r^2+1) = \frac{1}{4}(k+1)(k+2)((k+1)^2+(k+1)+2)$

$= \frac{1}{4}(k+1)(k+2)(k^2+3k+4)$

$$LHS = \sum_{r=1}^{k} r(r^2+1) + (k+1)((k+1)^2+1)$$

$$= \frac{1}{4}k(k+1)(k^2+k+2) + (k+1)(k^2+2k+2)$$

$$\frac{1}{4}(k+1)\left[k(k^2+k+2) + 4(k^2+2k+2)\right]$$

$$= \frac{1}{4}(k+1)\left[k^3+5k^2+10k+8\right]$$

$$= \frac{1}{4}(k+1)(k+2)(k^2+3k+4)$$

$$= RHS$$

$\therefore$ P_k is true $\Rightarrow$ P_{k+1} is true

$\because$ P_1 is true and P_k is true $\Rightarrow$ P_{k+1} is true,

$\therefore$ by Mathematical Induction, P_n is true $\forall n \in \mathbb{Z}^+$.

(ii) $u_0 = 2$ & $u_n = u_{n-1} + n^3 + n$ for $n \geqslant 1$

$u_1 = u_0 + 1 + 1 = 4$

$u_2 = u_1 + 2^3 + 2 = 14$

$u_3 = u_2 + 3^3 + 3 = 44$ #

(iii) $\sum_{r=1}^{n} (u_r - u_{r-1}) = \sum_{r=1}^{n} (r^3+r) = \sum_{r=1}^{n} r(r^2+1)$

$$\left.\begin{aligned} & u_1 - \cancel{u_0} \\ + & \cancel{u_2} - \cancel{u_1} \\ + & \cancel{u_3} - \cancel{u_2} \\ & \vdots \\ + & \cancel{u_{n-1}} - \cancel{u_{n-2}} \\ + & u_n - \cancel{u_{n-1}} \end{aligned}\right\} = \frac{1}{4}n(n+1)(n^2+n+2)$$

$$u_n - u_0 = \frac{1}{4}n(n+1)(n^2+n+2)$$

$$\therefore u_n = \frac{1}{4}n(n+1)(n^2+n+2) + 2 \ \#$$

Complex Numbers (I)

7. (a) $w^2 + (-1-8i)w + (-17+7i) = 0$

$(-1+5i)^2 + (-1-8i)(-1+5i) + (-7+7i)$

$= (1-10i-25) + (1-5i+8i+40) + (-17+7i)$

$= 0 \Rightarrow \therefore -1+5i$ is a root of the equation

$w^2 + (-1-8i)w + (-17+7i) = (w-(-1+5i))(w-\alpha)$

Compare constant term on both sides: $(-1+5i)\alpha = -17+7i$

$$\alpha = \frac{-17+7i}{-1+5i} \times \frac{-1-5i}{-1-5i}$$

$$= \frac{(17+35)+i(85-7)}{(-1)^2-(5i)^2}$$

$\therefore$ the second root $= 2+3i$

(b) $z^3 - 5z^2 + 16z + k = 0$

$(1+ai)^3 - 5(1+ai)^2 + 16(1+ai) + k = 0$

$1 + 3ai + 3(ai)^2 + (ai)^3 - 5(1+2ai+(ai)^2) + 16 + 16ai + k = 0$

Comparing real & imaginary parts on both sides:

$-1 - 3a^2 - 5 + 5a^2 + 16 + k = 0$ & $3a - a^3 - 10a + 16a = 0$

$12 + 2a^2 + k = 0$ $\qquad$ $a(a^2-9) = 0$

Sub $a=3$: $\qquad$ $a = 0$ (rejected $\because a > 0$) or $a = 3$ # or -3 (rejected $\because a > 0$)

$12 + 2(9) + k = 0$

$\therefore k = -30$ #

Maclaurin's Series

8. $y = f(x) = \tan(ax+b)$

$f'(x) = a\sec^2(ax+b) = a(1+\tan^2(ax+b))$

$= a(1+y^2) = a + ay^2$ (shown)

$f''(x) = a(2y\frac{dy}{dx}) = 2ay(a+ay^2)$

$= 2a^2(y+y^3)$ #

$f'''(x) = 2a^2(\frac{dy}{dx} + 3y^2\frac{dy}{dx})$

$= 2a^2(a(1+y^2) + 3y^2 a(1+y^2))$

$= 2a^3(1+y^2)(1+3y^2)$ #

When $b = \frac{\pi}{4}$, $y = \tan(ax + \frac{\pi}{4})$

When $x = 0$, $y = \tan\frac{\pi}{4} = 1 = f(0)$

$f'(0) = a(1+1) = 2a$

$f''(0) = 2a^2(1+1) = 4a^2$

$f'''(0) = 2a^3(2)(4) = 16a^3$

By Maclaurin's series, $y = 1 + 2ax + 4a^2\left(\frac{x^2}{2}\right) + 16a^3\left(\frac{x^3}{3!}\right) + \ldots$

$\approx 1 + 2ax + 2a^2x^2 + \frac{8}{3}a^3x^3$ #

(iii) $f(x) = \tan 2x \Rightarrow a = 2, b = 0$

When $x=0$, $y = f(0) = 0$

$f'(0) = 2(1) = 2$

$f''(0) = 2a^2(0) = 0$

$f'''(0) = 16$

$\therefore \tan 2x = 0 + 2x + 0x^2 + 16\left(\frac{x^3}{3!}\right) + \ldots$

$\approx 2x + \frac{8}{3}x^3$ #

Differential Equations

9(a) $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} = 10$, sub $y = \frac{dx}{dt}$

$\frac{dy}{dt} = \frac{d^2x}{dt^2}$

$\frac{dy}{dt} + 2y = 10$

$\therefore \frac{dy}{dt} = 10 - 2y$ (shown)

(b) $\int \frac{1}{10-2y}\,dy = \int dt$

$\frac{\ln|10-2y|}{-2} = t + c \Rightarrow \ln|10-2y| = -2t - 2c$

$|10-2y| = e^{-2t-2c} = e^{-2t} \cdot e^{-2c}$

$10 - 2y = \pm e^{-2c} e^{-2t} = Ae^{-2t}$, $A = \pm e^{-2c}$

$y = \frac{10 - Ae^{-2t}}{2} = 5 + Be^{-2t}$, $B = -\frac{A}{2}$

$\frac{dx}{dt} = 5 + Be^{-2t}$

When $t = 0$, $\frac{dx}{dt} = 0$, $0 = 5 + B \Rightarrow B = -5$

$\therefore y = 5 - 5e^{-2t}$ #

$\frac{dx}{dt} = 5(1 - e^{-2t})$

$x = 5\int (1 - e^{-2t})\,dt = 5\left(t - \frac{e^{-2t}}{-2}\right) + c$

When $t=0$, $x = 0$, $0 = 5\left(0 + \frac{1}{2}\right) + c \Rightarrow c = -\frac{5}{2}$

$\therefore x = 5\left(t + \frac{e^{-2t}}{2}\right) - \frac{5}{2}$ #

(ii) $\frac{d^2x}{dt^2} = 10 - 5\sin\frac{1}{2}t$

$\frac{dx}{dt} = \int 10 - 5\sin\frac{1}{2}t\,dt = 10t - \frac{5(-\cos\frac{1}{2}t)}{\frac{1}{2}} + c$

$= 10t + 10\cos\frac{1}{2}t + c$

When $t=0$, $\frac{dx}{dt} = 0$, $0 = 0 + 10 + c \Rightarrow c = -10$

$x = \int (10t + 10\cos\frac{1}{2}t - 10)\,dt = 5t^2 + 20\sin\frac{1}{2}t - 10t + d$

When $t=0$, $x=0$, $0 = 0 + d \Rightarrow d = 0$

$\therefore x = 5t^2 + 20\sin\frac{1}{2}t - 10t$ #

(iii) When $x = 5$, $5 = \left(t + \frac{e^{-2t}}{2}\right) - \frac{5}{2} \Rightarrow t = 1.47$ s for 1st model

$5 = 5t^2 + 20\sin\frac{1}{2}t - 10t \Rightarrow t = 1.05$ s for 2nd model.

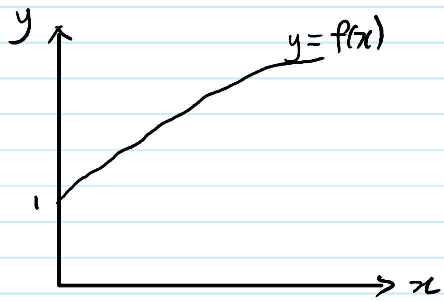
Functions

10.(a) $f : x \longrightarrow 1+\sqrt{x}, \quad x \in \mathbb{R}, \ x \geq 0$

(i) let $y = f(x) = 1+\sqrt{x}$

$x = (y-1)^2$

$\therefore f^{-1}(x) = (x-1)^2$ & $D_{f^{-1}} = R_f = [1, \infty)$ #



(ii) $ff(x) = f(1+\sqrt{x}) = 1+\sqrt{1+\sqrt{x}}$

$ff(x) = x \Rightarrow 1+\sqrt{1+\sqrt{x}} = x$ ——— (*)

$\Rightarrow 1+\sqrt{x} = (x-1)^2 = x^2-2x+1$

$\Rightarrow \sqrt{x} = x^2-2x$

$\Rightarrow x = (x^2-2x)^2 = x^4-4x^3+4x^2$

$\Rightarrow x^4-4x^3+4x^2-x = 0$

$\Rightarrow x(x^3-4x^2+4x-1) = 0$

$\therefore x = 0$ or $x^3-4x^2+4x-1 = 0$ (shown)

Using GC, $x = 2.618$ or 0.381966 or 1

$\because x \geq 2$ for eqn (*), $x = 2.618$ #

$ff(x) = x \Rightarrow f^{-1}ff(x) = f^{-1}(x)$

$\Rightarrow f(x) = f^{-1}(x)$

$\therefore x = 2.618$ satisfies the equation $f(x) = f^{-1}(x)$ #

(b) $g(n) = \begin{cases} 1, & n=0 \\ 2+g(\frac{1}{2}n), & n \text{ even} \\ 1+g(n-1), & n \text{ odd} \end{cases}$

(i) $g(4) = 2+g(2) = 2+(2+g(1))$

$= 4+(1+g(0)) = 5+1 = 6$ #

$g(7) = 1+g(6) = 1+(2+g(3)) = 3+(1+g(2))$

$= 4+(2+g(1)) = 6+(1+g(0)) = 8$ #

$g(12) = 2+g(6) = 2+(2+g(3)) = 4+(1+g(2))$

$= 5+(2+g(1)) = 7+(1+g(0)) = 9$ #

$g(5) = 1+g(4) = 1+6 = 7$

$g(6) = 2+g(3) = 2+(1+g(2)) = 3+(2+g(1)) = 5+(1+g(0)) = 7$

$\because g(5) = g(6) = 7$, g is not 1-1

$\therefore g^{-1}$ doesn't exist #

Vectors (II) + (III)

11.(a) When $a=0$, $p: \underset{\sim}{r} = \begin{pmatrix}1\\-3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\0\end{pmatrix} + \mu\begin{pmatrix}0\\4\\-2\end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$

$l: \underset{\sim}{r} = \begin{pmatrix}-1\\0\\1\end{pmatrix} + t\begin{pmatrix}-2\\1\\2\end{pmatrix}$, $t \in \mathbb{R}$

$\underset{\sim}{n} = \begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}0\\4\\-2\end{pmatrix} = \begin{pmatrix}-4\\2\\4\end{pmatrix} = 2\begin{pmatrix}-2\\1\\2\end{pmatrix} = 2 \times$ direction vector of l

$\therefore$ normal vector of p is parallel to direction vector of l,

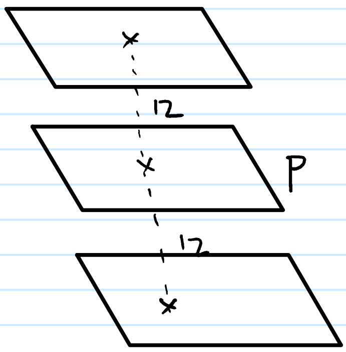
l is perpendicular to p.

Let $\begin{pmatrix}1\\-3\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\0\end{pmatrix} + \mu\begin{pmatrix}0\\4\\-2\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix} + t\begin{pmatrix}-2\\1\\2\end{pmatrix}$

$$\left.\begin{aligned} 1+\lambda &= -1-2t \\ -3+2\lambda+4\mu &= t \\ 2-2\mu &= 1+2t \end{aligned}\right\} \Rightarrow \begin{aligned} 2t+\lambda &= -2 \quad &\text{--- (1)} \\ -t+2\lambda+4\mu &= 3 \quad &\text{--- (2)} \\ 2t+2\mu &= 1 \quad &\text{--- (3)} \end{aligned}$$

Using GC, $t = -\frac{5}{9}$, $\lambda = -\frac{8}{9}$, $\mu = \frac{19}{18}$ #

(b) $p: \underset{\sim}{r}\cdot\begin{pmatrix}-2\\1\\2\end{pmatrix} = \begin{pmatrix}1\\-3\\2\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\2\end{pmatrix} = -2-3+4 = -1$



$\dfrac{\underset{\sim}{r}\cdot\begin{pmatrix}-2\\1\\2\end{pmatrix}}{\sqrt{2^2+1+2^2}} = \dfrac{-1}{3} =$ distance from origin to p

Equations of planes with distance 12 from p are

$\dfrac{\underset{\sim}{r}\cdot\begin{pmatrix}-2\\1\\2\end{pmatrix}}{3} = -\frac{1}{3} \pm 12 \Rightarrow \underset{\sim}{r}\cdot\begin{pmatrix}-2\\1\\2\end{pmatrix} = -1 \pm 36$

$\therefore -2x+y+2z = 35$ or $-2x+y+2z = 37$ #

(ii) $\underset{\sim}{n} = \begin{pmatrix}1\\2\\0\end{pmatrix} \times \begin{pmatrix}a\\4\\-2\end{pmatrix} = \begin{pmatrix}-4\\2\\4-2a\end{pmatrix}$

l do not intersect $p \Rightarrow l \,//\, p$

$\Rightarrow \begin{pmatrix}-4\\2\\4-2a\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\2\end{pmatrix} = 0$

$8+2+8-4a = 0$

$\therefore a = \frac{9}{4}$ #