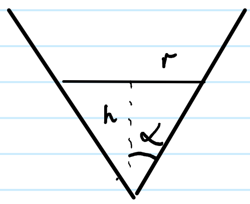
GCE A Level 2016 Paper 2

1. Applications of Differentiation



$\tan\alpha = 0.5$

$\frac{r}{h} = 0.5$

$r = 0.5h$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(0.5h)^2 h = \frac{1}{12}\pi h^3$

$\frac{dV}{dh} = \frac{1}{4}\pi h^2$

Given $V = 3 \Rightarrow \frac{1}{12}\pi h^3 = 3$

$h = \sqrt[3]{\frac{36}{\pi}}$

$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt}$

$0.1 = \frac{1}{4}\pi\left(\sqrt[3]{\frac{36}{\pi}}\right)^2\cdot\frac{dh}{dt} \Rightarrow \therefore \frac{dh}{dt} = 0.0251 \text{ m/min}$

2. Integration Techniques & Applications

(a)(i) $\int x^2\cos nx\,dx = x^2\left(\frac{\sin nx}{n}\right) - \int\frac{\sin nx}{n}(2x)\,dx$

$= \frac{x^2\sin nx}{n} - \frac{2}{n}\left[x\left(-\frac{\cos nx}{n}\right) - \int -\frac{\cos nx}{n}(1)\,dx\right]$

$= \frac{x^2\sin nx}{n} + \frac{2x}{n^2}\cos nx - \frac{2}{n^2}\left(\frac{\sin nx}{n}\right) + c$ #

(ii) $\int_{\pi}^{2\pi} x^2\cos nx\,dx = \left[\frac{x^2}{n}\sin nx + \frac{2x}{n^2}\cos nx - \frac{2}{n^3}\sin nx\right]_{\pi}^{2\pi}$

$= \left[\left(0 + \frac{4\pi}{n^2} - 0\right) - \left(0 + \frac{2\pi}{n^2}\cos n\pi - 0\right)\right]$

$= \frac{\pi}{n^2}(4 - 2\cos n\pi)$

$= \frac{\pi^2}{n^2}(4 \pm 2) = \frac{2\pi^2}{n^2}$ or $\frac{6\pi^2}{n^2} \Rightarrow a = 2$ or 6 #

(b) $V = \pi\int_0^2\left(\frac{x\sqrt{x}}{9-x^2}\right)^2 dx$, $u = 9 - x^2 = 9$ when $x = 0$, $= 5$ " $x = 2$

$= \pi\int_9^5\frac{(9-u)^{3/2}}{u^2}\left(-\frac{1}{2(9-u)^{1/2}}\,du\right)$

$\frac{du}{dx} = -2x$

$\frac{dx}{du} = -\frac{1}{2(9-u)^{1/2}}$

$= -\frac{\pi}{2}\int_9^5\frac{9-u}{u^2}\,du$

$= -\frac{\pi}{2}\int_9^5 9u^{-2} - \frac{1}{u}\,du$

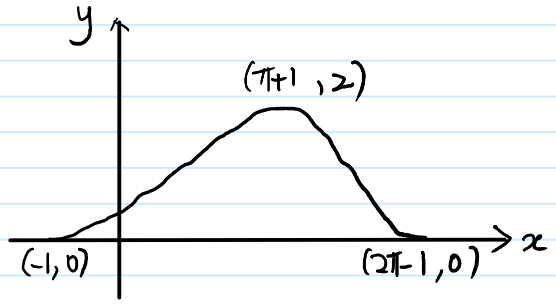
$= -\frac{\pi}{2}\left[\frac{9u^{-1}}{-1} - \ln|u|\right]_9^5 = -\frac{\pi}{2}\left[\left(-\frac{9}{5} - \ln 5\right) - (-1 - \ln 9)\right]$

$= -\frac{\pi}{2}\left[-\frac{4}{5} + \ln\frac{9}{5}\right] = \frac{\pi}{2}\left[\frac{4}{5} - \ln\frac{9}{5}\right]$ #

3. Applications of Differentiation + Integration

$x = t - \cos t, \quad y = 1 - \cos t, \quad 0 \le t \le 2\pi$

(i)



When $y = 1 - \cos t = 0$
$\cos t = 1$
$t = 0$ or 2π

When $t = 0$, $x = -1$
" $t = 2\pi$, $x = 2\pi - 1$

$\frac{dx}{dt} = 1 + \sin t, \quad \frac{dy}{dt} = \sin t$

$\frac{dy}{dx} = \frac{\sin t}{1 + \sin t} = 0 \Rightarrow \sin t = 0$
$t = 0, \pi$ or 2π

When $t = \pi$, $x = \pi + 1$, $y = 1 + 1 = 2$

(ii) $A = \int_{-1}^{a - \cos a} y \, dx$

$= \int_0^a (1 - \cos t)(1 + \sin t) \, dt$

$= \int_0^a (1 - \cos t + \sin t - \sin t \cos t) \, dt$

$= \int_0^a (1 - \cos t + \sin t - \frac{1}{2} \sin 2t) \, dt$

$= \left[t - \sin t - \cos t + \frac{\cos 2t}{4} \right]_0^a$

$= \left(a - \sin a - \cos a + \frac{\cos 2a}{4} \right) - \left(-1 + \frac{1}{4} \right)$

$= a + \frac{3}{4} - \sin a - \cos a + \frac{\cos 2a}{4}$ #

(iii) When $t = \frac{\pi}{2}$, $x = \frac{\pi}{2}$, $y = 1$, $\frac{dy}{dx} = \frac{1}{2}$

Equation of normal where $t = \frac{\pi}{2}$: $y - 1 = -\frac{1}{(\frac{1}{2})} \left(x - \frac{\pi}{2} \right)$

$\therefore y = -2x + \pi + 1$

When $x = 0$, $y = \pi + 1 \Rightarrow F = (0, \pi + 1)$
" $y = 0$, $2x = \pi + 1 \Rightarrow E = \left(\frac{\pi + 1}{2}, 0 \right)$

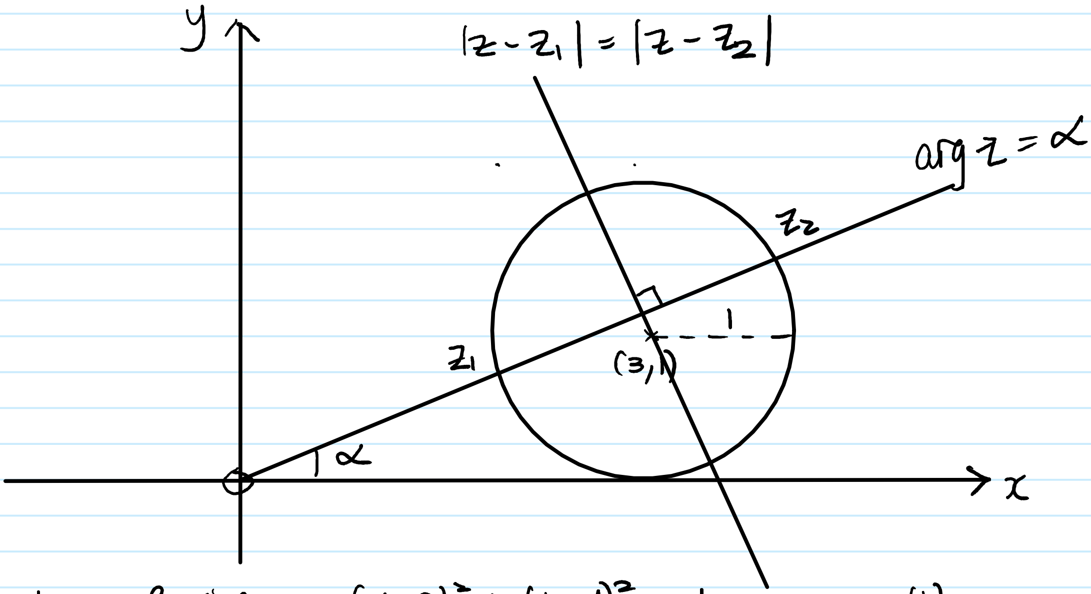
$\therefore$ Area of $\triangle OEF = \frac{1}{2} \left(\frac{\pi + 1}{2} \right) (\pi + 1)$

$= \frac{1}{4} (\pi + 1)^2$ unit^2

Complex Numbers

4(a) $|z-(3+i)| = 1$ & $\arg z = \alpha = \tan^{-1} 0.4$

(i)



(ii) Equation of circle : $(x-3)^2 + (y-1)^2 = 1$ —— (1)

" " perpendicular bisector : $y-1 = -\frac{1}{0.4}(x-3)$ —— (2)

Sub (2) into (1) : $(x-3)^2 + 2.5^2(x-3)^2 = 1$

$(x-3)^2 = \frac{4}{29}$

$x = \pm\frac{2}{\sqrt{29}} + 3 = 3.37139$ or 2.62861

$y = \cancel{0}.071525$ or 1.92848

$\therefore z = 3.37 + 0.0715i$ or $2.63 + 1.93i$

(b) (i) $w = 2-2i = \sqrt{8}e^{i(-\frac{\pi}{4})}$

Let $z = w^{\frac{1}{3}} \Rightarrow z^3 = \sqrt{8}e^{i(-\frac{\pi}{4})} = \sqrt{8}e^{i(-\frac{\pi}{4}+2k\pi)}$, $k \in \mathbb{Z}$

$z = 8^{\frac{1}{6}}e^{i(-\frac{\pi}{12}+\frac{2k\pi}{3})}$, $k = 0, \pm 1$

$= 8^{\frac{1}{6}}e^{i(-\frac{\pi}{12})}$, $8^{\frac{1}{6}}e^{i(\frac{7\pi}{12})}$, $8^{\frac{1}{6}}e^{i(-\frac{3\pi}{4})}$

(ii) $\arg(w^* w^n) = -\arg w + n\arg w = \frac{\pi}{2}$

$\frac{\pi}{4} + n(-\frac{\pi}{4}) = \frac{\pi}{2}, \frac{\pi}{2} \pm 2\pi, \ldots$

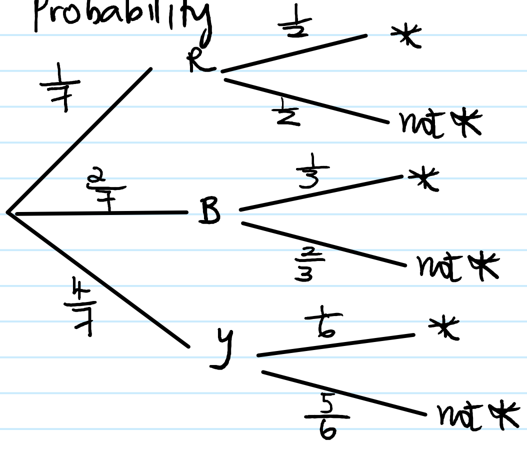
$(-\frac{\pi}{4})n = \frac{\pi}{4}, \frac{9\pi}{4}, -\frac{7\pi}{4}, \ldots$

$n = -1, -9, 7, \ldots$

$\therefore$ smallest +ve whole no. of $n = 7_{\#}$

Probability

5.



(i) P(player wins game)

$= (\frac{1}{7})(\frac{1}{2}) + (\frac{2}{7})(\frac{1}{3}) + (\frac{4}{7})(\frac{1}{6})$

$= \frac{11}{42}$ #

(ii) P(B | player wins game)

$= \frac{P(\text{B} \cap \text{player wins game})}{P(\text{player wins game})}$

$= \frac{(\frac{2}{7})(\frac{1}{3})}{\frac{11}{42}} = \frac{4}{11}$ #

(iii) P(player wins 3 games, each time different colour)

$= (\frac{1}{7})(\frac{1}{2}) \times (\frac{2}{7})(\frac{1}{3}) \times (\frac{4}{7})(\frac{1}{6}) \times 3! = \frac{4}{1029}$ #

Sampling Methods

6(i) (a) No. of males in sample $= \frac{2345 + 1013 + 237 + 344}{6599} \times 100$

$= \frac{3939}{6599} \times 100 \approx 60$

(b) No. of females from Development dept. $= \frac{679}{6599} \times 100$

≈ 10

(ii) The stratified sample from part (i) should not be used for hypothesis test as the sample is based on department and gender and not based on age.

(ii) Hypothesis Testing

To test $H_0: \mu = 37$

against $H_1: \mu < 37$

Managing Director's belief should be accepted

$\Rightarrow$ Reject H_0

$\Rightarrow$ $Z_{calc} < Z_{critical}$

$$\frac{\bar{x} - 37}{\sqrt{140/80}} < -1.64485$$

$$\bar{x} < 34.824$$

$\therefore$ Set of possible values of $\bar{x} = \{\bar{x} \in \mathbb{R} : \bar{x} < 34.8\}$

(iv) $\bar{x} = 35.2$

Using GC, p-value $= 0.0868087$

Managing Director's belief should not be accepted

$\Rightarrow$ Do not reject H_0

$\Rightarrow$ $p > \alpha$

$$\alpha\% < 0.0868087 \times 100\%$$

$\therefore$ Set of possible values of $\alpha = \{\alpha \in \mathbb{R} : \alpha < 8.68\}$

Permutations + Combinations

7. 6 men, 6 women

(i) No. of ways they are all women $= \binom{4}{3} \times 3! = 24$ #

(ii) No. of ways ≥ 1 woman + ≥ 1 man

$= \binom{10}{3} \times 3! - 24 - \binom{6}{3} \times 3! = 576$ #

(iii) P(chairperson, secretary + treasurer in adjacent places)

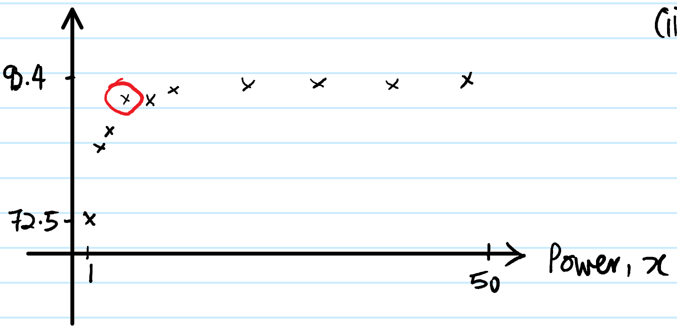
$= \dfrac{(8-1)! \times 3!}{(10-1)!} = \dfrac{1}{12}$ #

(iv) P(chairperson, secretary + treasurer separated by ≥ 1 person)

$= \dfrac{(7-1)! \times 7 \times 6 \times 5}{(10-1)!} = \dfrac{5}{12}$ #

Correlation + Regression

8(i) Efficiency, y %



(ii) Relationship between x + y should not be modelled by $y = ax + b$ since as x increases, y increases at a decreasing rate as shown by the scatter diagram.

(iii) $y = \frac{c}{x} + d \Rightarrow$ when $x \to \infty$, $y \to d$

$\therefore$ d is positive as it represents the maximum efficiency as the power increases.

c is negative since $\frac{1}{x}$ is increasing $\Rightarrow$ x is decreasing + y is also decreasing.

(iv) $r = -0.979546 \approx -0.980$

$c = -17.48360 \approx -17.5$

$d = 91.750109 \approx 91.8$ #

(v) When $x = 3$, $y = \dfrac{-17.48360}{3} + 91.750109$

$= 85.92224 \approx 85.9$ #

This estimate is reliable $\because$ $x = 3$ lies within the given data range and $r = -0.980$ is closed to -1 which shows a strong negative linear correlation.

Normal Distribution

9(a) $X \sim N(15, \sigma^2)$, $P(10 < X < 20) = 0.5$

$$P(X < 10) = P\left(Z < \frac{10-15}{\sigma}\right) = 0.25$$

$$\frac{-5}{\sigma} = -0.6744897$$

$$\therefore \sigma = 7.41301 \approx 7.41 \#$$

(b) $Y \sim B(4, p)$, $P(Y=1) + P(Y=2) = 0.5$

$$\binom{4}{1} p(1-p)^3 + \binom{4}{2} p^2(1-p)^2 = 0.5$$

$$4p(1-3p+3p^2-p^3) + 6p^2(1-2p+p^2) = 0.5$$

$$4p - 12p^2 + \cancel{12p^3} - 4p^4 + 6p^2 - \cancel{12p^3} + 6p^4 = 0.5$$

$$2p^4 - 6p^2 + 4p = 0.5$$

$$\therefore 4p^4 - 12p^2 + 8p = 1 \# \text{ (shown)}$$

Using GC, $p = 0.5994456$ or 0.16591

$$\therefore p \approx 0.599 \text{ or } 0.166 \#$$

(c) Let W be no of right guesses out of 100 questions

$W \sim B(100, \frac{1}{3})$

$\because n = 100$ is large, $np = 100(\frac{1}{3}) > 5$, $n(1-p) = 100(\frac{2}{3}) > 5$,

$W \sim N(\frac{100}{3}, \frac{200}{9})$ approx.

$$P(W \geq 30) \overset{c.c.}{\cong} P(W \geq 29.5)$$

$$= 0.79194 \approx 0.792 \#$$

Poisson Distribution & Approximation

10(i) The mean no. of a particular type of weeds is constant in Mia's field. The particular type of weeds found in Mia's field is independent of each other.

(ii) Let X be no. of dandelion plants in $1\,m^2$ of Mia's field.

$X \sim Po(1.5)$

$P(X \geq 2) = 1 - P(X \leq 1)$

$= 0.44217 \approx 0.442$ #

(iii) Let Y be no. of dandelion plants in $4m^2$ of Mia's field.

$Y \sim Po(1.5 \times 4)$

$P(Y \leq 3) = 0.1512 \approx 0.151$ #

(iv) Let W be no. of dandelion plants in $80\,m^2$ of Mia's field.

$W \sim Po(1.5 \times 80) = Po(120)$

$\because \lambda = 120 > 10$, $W \sim N(120, 120)$ approx.

$P(110 \leq W \leq 140) \overset{c.c.}{=} P(109.5 \leq W \leq 140.5)$

$= 0.80045 \approx 0.800$ #

(v) Let D be no. of daisies in $1\,m^2$ of Mia's field.

E " " " " in $2\,m^2$ of "

$D \sim Po(\lambda)$, $E \sim Po(2\lambda)$

$P(D \leq 2) = P(E > 2)$

$e^{-\lambda} + e^{-\lambda}\lambda + \dfrac{e^{-\lambda}\lambda^2}{2!} = 1 - P(E \leq 2)$

$= 1 - \left(e^{-2\lambda} + e^{-2\lambda}(2\lambda) + \dfrac{e^{-2\lambda}(2\lambda)^2}{2}\right)$

Using GC, $\lambda = 1.85433 \approx 1.85$ #