GCE A Level 2017 Paper 1

1. Maclaurin's Series

$$e^{2x}\ln(1+ax) = \left(1+2x+\frac{(2x)^2}{2}+\frac{(2x)^3}{3!}+\ldots\right)\left(ax-\frac{(ax)^2}{2}+\frac{(ax)^3}{3}+\ldots\right)$$
$$\approx \left(1+2x+2x^2+\tfrac{4}{3}x^3\right)\left(ax-\frac{a^2x^2}{2}+\frac{a^3x^3}{3}\right)$$
$$\approx ax-\frac{a^2x^2}{2}+\frac{a^3x^3}{3}+2ax^2-a^2x^3+2ax^3$$
$$= ax+\left(-\frac{a^2}{2}+2a\right)x^2+\left(\frac{a^3}{3}-a^2+2a\right)x^3$$

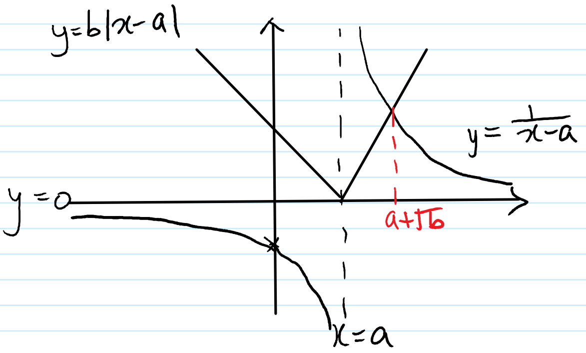
No term in x^2 $\Rightarrow$ $-\frac{a^2}{2}+2a=0$

$a\left(2-\frac{a}{2}\right)=0$

$a=0$ or $a=4$

(rejected $\because a\neq 0$)

2. Inequalities / Curve sketching



When $\frac{1}{x-a}=b(x-a)$

$b(x-a)^2=1$

$(x-a)^2=\frac{1}{b}$

$x-a=\pm\frac{1}{\sqrt{b}}$

$x=a\pm\frac{1}{\sqrt{b}}$

$$\frac{1}{x-a}<b|x-a| \Rightarrow x<a \text{ or } x>a+\sqrt{b}$$

3. $y^2-2xy+5x^2-10=0$ —— ①

$2y\frac{dy}{dx}-2\left[x\frac{dy}{dx}+y\right]+10x=0$ —— ②

At stationary pts, $\frac{dy}{dx}=0$

$-2y+10x=0$

$y=5x$

Sub $y=5x$ into equation ①:

$(5x)^2-2x(5x)+5x^2-10=0$

$25x^2-10x^2+5x^2=10$

$20x^2=10 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$

Differentiate ② wrt x:

$$2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2-2\left[\left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}\right)+\frac{dy}{dx}\right]+10=0$$

When $x=\frac{1}{\sqrt{2}}$, $\frac{dy}{dx}=0$, $y=5\left(\frac{1}{\sqrt{2}}\right)=\frac{5}{\sqrt{2}}$

$$2\left(\frac{5}{\sqrt{2}}\right)\frac{d^2y}{dx^2}+2(0)-2\left[\frac{1}{\sqrt{2}}\frac{d^2y}{dx^2}+0\right]=-10$$

$$\frac{8}{\sqrt{2}}\frac{d^2y}{dx^2}=-10 \Rightarrow \frac{d^2y}{dx^2}=\frac{-10}{\left(\frac{8}{\sqrt{2}}\right)}<0$$

$\Rightarrow$ it is a maximum

4. $y = \dfrac{4x+9}{x+2} \Rightarrow \dfrac{dy}{dx} = \dfrac{(x+2)(4)-(4x+9)}{(x+2)^2}$

$= \dfrac{-1}{(x+2)^2}$

$\because (x+2)^2 \geqslant 0 \ \forall x \in \mathbb{R}, \ \dfrac{dy}{dx} < 0$

$y = \dfrac{4(x+2)+1}{x+2} = 4 + \dfrac{1}{x+2}$

Equations of asymptotes : $y = 4$ and $x = -2$

$y = \dfrac{1}{x} \xrightarrow{x \to x+2}$ $y = \dfrac{1}{x+2}$

Translation along -ve x-axis by 2 units

$\downarrow$ $y \to y-4$

Translation along the y-axis by 4 units

$y = 4 + \dfrac{1}{x+2}$

5. System of linear Equations / Equation of tangent

Let $f(x) = x^3 + ax^2 + bx + c$

$f(1) = 1 + a + b + c = 8 \Rightarrow a+b+c = 7$ — ①

$f(2) = 8 + 4a + 2b + c = 12 \Rightarrow 4a+2b+c = 4$ — ②

$f(3) = 27 + 9a + 3b + c = 25 \Rightarrow 9a+3b+c = -2$ — ③

Using GC, $a = -\frac{3}{2}$, $b = \frac{3}{2}$, $c = 7$

$f(x) = x^3 - \frac{3}{2}x^2 + \frac{3}{2}x + 7$

$f'(x) = 3x^2 - 3x + \frac{3}{2}$

$= 3\left[x^2 - x + \frac{1}{2}\right]$

$= 3\left[(x-\frac{1}{2})^2 - \frac{1}{4} + \frac{1}{2}\right]$

$= 3\left[(x-\frac{1}{2})^2 + \frac{1}{4}\right]$

$\because (x-\frac{1}{2})^2 \geqslant 0 \ \forall x \in \mathbb{R}, \ f'(x) > 0$

$\because f'(x) > 0 \ \forall x \in \mathbb{R}$, f is strictly increasing and will cut the x-axis at only 1 point,

$\therefore f(x) = 0$ has only 1 real root.

Using GC, when $f(x) = 0$,

$x = -1.33$ (3 s.f.)

(ii) Tangent is // $y = 2x - 3 \Rightarrow f'(x) = 2$

$3x^2 - 3x + \frac{3}{2} = 2$

$3x^2 - 3x - \frac{1}{2} = 0$

$x = -0.145497$ or 1.145497224

≈ -0.145 or 1.15 (3 s.f.)

6. Vectors

(i) Straight line passing thru' fixed point with position vector $\underset{\sim}{a}$ and parallel to vector $\underset{\sim}{b}$.

(ii) Plane perpendicular to vector $\underset{\sim}{n}$ with distance d from origin.
d represents the perpendicular distance from origin to plane.

(iii) Sub $\underset{\sim}{r} = \underset{\sim}{a} + t\underset{\sim}{b}$ into $\underset{\sim}{r}\cdot\underset{\sim}{n} = d$,

$$(\underset{\sim}{a}+t\underset{\sim}{b})\cdot\underset{\sim}{n} = d \Rightarrow \underset{\sim}{a}\cdot\underset{\sim}{n} + t\,\underset{\sim}{b}\cdot\underset{\sim}{n} = d$$

$$t = \frac{d - \underset{\sim}{a}\cdot\underset{\sim}{n}}{\underset{\sim}{b}\cdot\underset{\sim}{n}}$$

$$\therefore \underset{\sim}{r} = \underset{\sim}{a} + \left(\frac{d-\underset{\sim}{a}\cdot\underset{\sim}{n}}{\underset{\sim}{b}\cdot\underset{\sim}{n}}\right)\underset{\sim}{b}$$

$\because$ $\underset{\sim}{b}\cdot\underset{\sim}{n} \neq 0 \Rightarrow$ line is not parallel to plane,
$\therefore$ the solution represents the point of intersection between the line and the plane.

7. Integration Techniques

(i) $$\int \sin(2mx)\sin(2nx)\,dx = -\frac{1}{2}\int \cos(2mx+2nx) - \cos(2mx-2nx)\,dx$$

$$= -\frac{1}{2}\left[\frac{\sin(2m+2n)x}{2m+2n} - \frac{\sin(2m-2n)x}{2m-2n}\right] + c$$

$$= -\frac{1}{4}\left[\frac{\sin(2m+2n)x}{m+n} - \frac{\sin(2m-2n)x}{m-n}\right] + c$$

(ii) $$\int_0^{\pi} (f(x))^2\,dx = \int_0^{\pi} [\sin(2mx)\sin(2nx)]^2\,dx$$

$$= \int_0^{\pi} \sin^2(2mx) + 2\sin(2mx)\sin(2nx) + \sin^2(2nx)\,dx$$

$$= \int_0^{\pi} \frac{1-\cos(4mx)}{2} + \frac{1-\cos(4nx)}{2} + 2\sin(2mx)\sin 2nx\,dx$$

$$= \left[\frac{x}{2} - \frac{\sin 4mx}{2(4m)} + \frac{x}{2} - \frac{\sin(4nx)}{2(4n)} + 2\left[-\frac{1}{4}\left(\frac{\sin(2m+2n)x}{m+n} - \frac{\sin(2m-2n)x}{m-n}\right]\right]\right]_0^{\pi}$$

$$= \pi - 0 = \pi$$

8. Complex Numbers

$$z^2(1-i) - 2z + 5 + 5i = 0$$

$$z = \frac{2 \pm \sqrt{4 - 4(1-i)(5+5i)}}{2(1-i)}$$

$$= \frac{2\pm\sqrt{4-40}}{2(1-i)} = \frac{2\pm 6i}{2(1-i)}$$

$$= \frac{1+3i}{1-i}\times\frac{1+i}{1+i} \quad \text{or} \quad \frac{1-3i}{1-i}\times\frac{1+i}{1+i}$$

$$= -1+2i \quad \text{or} \quad 2-i$$

(b)(i) $$w = 1-i$$

$$w^2 = (1-i)^2 = 1-2i-1 = -2i$$

(b)(i) $w = 1-i$

$w^2 = (1-i)^2 = 1-2i-1 = -2i$

$w^3 = (-2i)(1-i) = -2-2i$

$w^4 = (-2i)^2 = -4$

$w^4 + pw^3 + 39w^2 + qw + 58 = 0$

$-4 + p(-2-2i) + 39(-2i) + q(1-i) + 58 = 0$

Comparing real and imaginary parts :

$-4 - 2p + q + 58 = 0$ & $-2p - 78 - q = 0$

$-2p + q = -54$ — ① $-2p - q = 78$ — ②

① + ② : $-4p = 24 \Rightarrow p = -6$

From ① : $q = -54 + 2(-6) = -66$

(ii) $w^4 - 6w^3 + 39w^2 - 66w + 58 = 0$

$\because$ all coefficients are real, $w = 1+i$ is also a root

$[w-(1-i)][w-(1+i)][w^2 + aw + b] = 0$

$[(w-1)^2 - (i)^2][w^2 + aw + b] = 0$

$(w^2 - 2w + 2)(w^2 + aw + b) = 0$

Comparing w^3 term : $-6 = a - 2 \Rightarrow a = -4$

" constant : $58 = 2b \Rightarrow b = 29$

$\therefore w^4 - 6w^3 + 39w^2 - 66w + 58 = (w^2 - 2w + 2)(w^2 - 4w + 29)$

9. Σ Notation / Method of Difference

(i) $S_n = An^2 + Bn \Rightarrow S_{n-1} = A(n-1)^2 + B(n-1)$

$u_n = S_n - S_{n-1}$

$= An^2 + Bn - [A(n-1)^2 + B(n-1)]$

$= An^2 + \cancel{Bn} - [A(n^2 - 2n + 1) + \cancel{Bn} - B]$

$= 2An - A + B$

(ii) $u_{10} = 48 \Rightarrow 20A - A + B = 48$

$19A + B = 48$ — ①

$u_{17} = 90 \Rightarrow 34A - A + B = 90$

$33A + B = 90$ — ②

② − ① : $14B = 42 \Rightarrow A = 3$

From ①: $B = 48 - 57 = -9$

(b) $r^2(r+1)^2 - (r-1)^2 r^2 = r^2(r^2 + 2r + 1) - (r^2 - 2r + 1)r^2$

From ①:

(b) $r^2(r+1)^2 - (r-1)^2 r^2 = r^2(r^2+2r+1) - (r^2-2r+1)r^2$

$= 2r^3 + 2r^3 = 4r^3 \quad \therefore k=4$

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}\sum_{r=1}^{n} r^2(r+1)^2 - (r-1)^2 r^2$$

$$\begin{aligned} = \tfrac{1}{4}\Big[& (1)(2)^2 - 0 \\ & + 2^2(3)^2 - 1^2(2)^2 \\ & + 3^2(4)^2 - 2^2(3)^2 \\ & \vdots \\ & + (n-2)^2(n-1)^2 - (n-3)^2(n-2)^2 \\ & + (n-1)^2(n^2) - (n-2)^2(n-1)^2 \\ & + (n^2)(n+1)^2 - (n-1)^2 n^2 \Big] \end{aligned}$$

$$= \tfrac{1}{4}\left[n^2(n+1)^2 \right]$$

(c) $\displaystyle\sum_{r=0}^{\infty} \frac{x^r}{r!} = \sum_{r=0}^{\infty} a_r \Rightarrow a_r = \frac{x^r}{r!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right|$$

$$= 0 \quad \forall\, x \in \mathbb{R}$$

$$< 1 \Rightarrow \sum_{r=0}^{\infty}\frac{x^r}{r!} \text{ converges}$$

$$\sum_{r=0}^{\infty}\frac{x^r}{r!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= e^x$$

10. Vectors

(i) $l_c : \underset{\sim}{r} = \lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}, \quad \lambda \in \mathbb{R}$

$\overrightarrow{PQ} = \begin{pmatrix}5\\7\\a\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}4\\5\\a+1\end{pmatrix}$

$l_{PQ} : \underset{\sim}{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}4\\5\\a+1\end{pmatrix}, \quad \mu \in \mathbb{R}$

Let $\lambda\begin{pmatrix}3\\1\\-2\end{pmatrix} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \mu\begin{pmatrix}4\\5\\a+1\end{pmatrix}$

$3\lambda = 1 + 4\mu$
$\lambda = 2 + 5\mu$ $\Rightarrow$ $\left.\begin{aligned} 3\lambda - 4\mu &= 1 \\ \lambda - 5\mu &= 2 \end{aligned}\right\}$ Using GC, $\mu = -\frac{5}{11}, \lambda = -\frac{3}{11}$

$-2\lambda = -1 + \mu(a+1)$

$$\lambda = 2 + 5\mu \quad \Rightarrow \quad \lambda - 5\mu = 2 \Big\} \quad \mu = -\tfrac{5}{11},\ \lambda = -\tfrac{3}{11}$$

$$-2\lambda = -1 + \mu(a+1)$$

$$-2\left(-\tfrac{3}{11}\right) = -1 + \left(-\tfrac{5}{11}\right)(a+1) \Rightarrow a = -4.4$$

(ii) R lies on $C \Rightarrow \overrightarrow{OR} = \lambda\begin{pmatrix}3\\1\\-2\end{pmatrix}$ for some $\lambda \in \mathbb{R}$

$$\overrightarrow{PR} = \lambda\begin{pmatrix}3\\1\\-2\end{pmatrix} - \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}3\lambda-1\\ \lambda-2\\ -2\lambda+1\end{pmatrix}$$

$$\overrightarrow{QR} = \lambda\begin{pmatrix}3\\1\\-2\end{pmatrix} - \begin{pmatrix}5\\7\\-3\end{pmatrix} = \begin{pmatrix}3\lambda-5\\ \lambda-7\\ -2\lambda+3\end{pmatrix}$$

$$\overrightarrow{PR}\cdot\overrightarrow{QR} = \begin{pmatrix}3\lambda-1\\ \lambda-2\\ -2\lambda+1\end{pmatrix}\cdot\begin{pmatrix}3\lambda-5\\ \lambda-7\\ -2\lambda+3\end{pmatrix}$$

$$= (3\lambda-1)(3\lambda-5) + (\lambda-2)(\lambda-7) + (-2\lambda+1)(-2\lambda+3)$$

$$= 9\lambda^2 - 18\lambda + 5 + \lambda^2 - 9\lambda + 14 + 4\lambda^2 - 8\lambda + 3$$

$$= 14\lambda^2 - 35\lambda + 22$$

$$= 14\left[\lambda^2 - \tfrac{35}{14}\lambda + \tfrac{22}{14}\right] = 14\left[\lambda^2 - \tfrac{5}{2}\lambda + \tfrac{11}{7}\right]$$

$$= 14\left[\left(\lambda - \tfrac{5}{4}\right)^2 - \left(\tfrac{5}{4}\right)^2 + \tfrac{11}{7}\right]$$

$$= 14\left[\left(\lambda - \tfrac{5}{4}\right)^2 + \tfrac{1}{112}\right]$$

$$\neq 0 \Rightarrow \angle PRQ \text{ cannot be } 90^\circ$$

(iii)
$$|\overrightarrow{PR}| = \sqrt{(3\lambda-1)^2 + (\lambda-2)^2 + (-2\lambda+1)^2}$$

$$= \sqrt{(9\lambda^2 - 6\lambda + 1) + (\lambda^2 - 4\lambda + 4) + (4\lambda^2 - 4\lambda + 1)}$$

$$= \sqrt{14\lambda^2 - 14\lambda + 6} = \sqrt{14\left(\lambda^2 - \lambda + \tfrac{6}{14}\right)}$$

$$= \sqrt{14\left[\left(\lambda - \tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2 + \tfrac{3}{7}\right]} = \sqrt{14\left[\left(\lambda - \tfrac{1}{2}\right)^2 + \tfrac{5}{28}\right]}$$

$|\overrightarrow{PR}|$ is min when $\lambda = \tfrac{1}{2}$

$$\Rightarrow \overrightarrow{OR} = \tfrac{1}{2}\begin{pmatrix}3\\1\\-2\end{pmatrix} \quad \therefore R = \left(\tfrac{3}{2}, \tfrac{1}{2}, -1\right)$$

& $\min |\overrightarrow{PR}| = \sqrt{14\left(\tfrac{5}{28}\right)} = \sqrt{\tfrac{5}{2}}$

11. Differential Equations

(i)(a) $\dfrac{dv}{dt} = c$

(b) $v = \int c\, dt = ct + d$, d = constant

when $t = 0$, $v = 4$, $4 = d$

" $t = 2.5$, $v = 29$, $29 = 2.5c + 4 \Rightarrow c = 10$

$\therefore v = 10t + 4$

(ii) $\dfrac{dv}{dt} = 10 - kv$

$$\int \frac{1}{10 - kv}\, dv = \int 1\, dt$$

$$\int \frac{1}{10-kv}\,dv = \int 1\,dt$$

$$\frac{\ln|10-kv|}{-k} = t + c$$

$$|10-kv| = e^{-kt-kc} = e^{-kt}\cdot e^{-kc}$$

$$10-kv = \pm e^{-kc}\cdot e^{-kt}$$

$$= Ae^{-kt}, \quad A = \pm e^{-kc}$$

When $t=0$, $v=0$, $10 = A$

$$10-kv = 10e^{-kt} \Rightarrow v = \frac{10-10e^{-kt}}{k}$$

(iii) When $t \to \infty$, $v \to 40$, $40 = \frac{10}{k} \Rightarrow k = \frac{1}{4}$

$$v = 40\left(1-e^{-\frac{t}{4}}\right)$$

When $v = 0.9(40) = 36$,

$$36 = 40\left(1-e^{-\frac{t}{4}}\right)$$

$$e^{-\frac{t}{4}} = 1 - \frac{9}{10} = \frac{1}{10}$$

$$-\frac{t}{4} = \ln\frac{1}{10} \Rightarrow t = 9.21\text{ s.}$$